Percolation Theory

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Contents

Lecture 1

Accoarding dictionary: Percolation is the flow of a liquid through a porous medium. For example, coffee machine and vulcanic rocks.

Definitions and notation

$\mathbb{Z}^d$. Each length one line segment between to points, are "open" with probability P and "closed" with probability $1 - P$.
PERCOLATION there exists infinite path of open edges (="bonds")
Question: For what values of, P is percolation possible.
CRITICAL PROBABILITY: $p_c = \inf \{p \in [0,1] : \mathbb{P}_p(\text{percolation}) > 0\}$.
If $p < p_c$ then $\mathbb{P}_p(\text{Perc}) = 0$.

Exercise show if $d = 1$ then $p_c = 1$.

$d = 2$,has already rich behaviour. Conjecture (Hammfrey): $p_c = 1/2$ in $d = 2$. (This is also proven)

Graph $G = (V, E)$ where V vertices/points, E connections/edges/bonds.
$\mathbb{Z}^d$ denotes graph $(\mathbb{Z}^d, \{\{u, v\} \in \mathbb{Z}^d \times \mathbb{Z}^d : \|u - v\| = 1\}$.
$v \in V$ incident with $e \in E$ if $e = uv$ for some $u \in V$.
$G \subseteq H$ where G, H graphs if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.
WALK in G: $V_0, \ldots, V_k$ s.t. $V_i \in E(G)$
PATH walk with all vertices distinct.
CLOSED WALK walk $V_0 = V_n$.
CYCLE/CIRCUIT closed walk with $V_1, \ldots, V_n$ distinct.
CONNECTED COMPONENT=CLUSTER in G is maximal connected subgraph.
Where CONNECTED $\exists$ path between any 2 points, and MAXIMAL If i add any vertex not yet there, it fails to be connected.

$\{a \leftrightsquigarrow b\} = \{\exists \text{ (open) path between } a \& b\}$
$\{a \leftrightsquigarrow \infty\} = \{\exists \infty \text{path starting at } a\}$
$A \leftrightsquigarrow B$ if $A, B \subseteq V$ s.t. $\exists a \in A, b \in B$ s.t. $a \leftrightsquigarrow b$.
$a \overset{C}{\leftrightsquigarrow} b = \{\exists a, b \text{ path that stays inside set } C\}$.
$\Lambda_n := \{-n, n\}^d$ and $\partial\Lambda_n = \{z \in \mathbb{Z}^d : |z_i|_n, \forall n \& \exists i : |z_i| = n\}$.
We can describe percolation model via a sequence of coin flips.
$(X_e : e \in E)$ I.I.D.$\overset{d}{=} Be(p)$. Here $\overset{d}{=}$ means distributed like, IID means independent

identically distributed, $Be(p)$ means that $X = \begin{cases} 1, \text{with probability } p \\ 0, \text{with probability } 1-p \end{cases}$.

Fix enumerations $e_1, e_2, \ldots$ of $E(\mathbb{Z}^d)$ and $X_{e_1}, X_{e_2}, \ldots$ is IID that contains all information on the model.
$\Omega = \{0,1\}^{E(\mathbb{Z}^d)}$ configuration space $(X_e : e \in E(\mathbb{Z}^d)) \in \Omega$.
Event correspond to a set $A \subseteq \Omega$

Percolation function

PERCOLATION FUNCTION: $\theta(p) := \mathbb{P}_p(0 \leftrightsquigarrow \infty)$.
Lemma:
$\theta(p) > 0 \Leftrightarrow \mathbb{P}_p(\text{percolation}) > 0$
Proof part 1:
$\{0 \leftrightsquigarrow \infty\} \subseteq \{\text{percolation}\}$. So $\theta(p) = \mathbb{P}(0 \leftrightsquigarrow \infty) \subseteq \mathbb{P}(\text{percolation})$. So $\mathbb{P}(\text{percolation}) = 0 \Rightarrow \theta(p) = 0$.
Proof part 2:
Note $\forall z \in \mathbb{Z}^d, \mathbb{P}(z \leftrightsquigarrow \infty) = \mathbb{P}(0 \leftrightsquigarrow \infty)$ this is because there is nothing special about the origin.

$$\mathbb{P}(\text{perc}) = \mathbb{P}\left(\bigcup_{z \in \mathbb{Z}^d} \{z \leftrightsquigarrow \infty\}\right) \leq \sum_{z \in \mathbb{Z}^n} \mathbb{P}(z \leftrightsquigarrow \infty) = \sum_{z \in \mathbb{Z}^d} \theta(p)$$

So if $\theta(p) = 0 \Rightarrow P(\text{perc}) = 0$. $\leq$ follows from countable SEE PROB NOTES
Therefore we can also say $p_c = \inf\{p : \theta(p) > 0\}$.

Up-set, down-set, coupling

$A \subseteq \Omega = \{0,1\}^{E(\mathbb{Z}^d)}$ is an UP-SET (INCREASING) if for every $a = (a_e)_{e \in E(\mathbb{Z}^d)} \in A$ and every $e \in E(\mathbb{Z}^d)$ the vector $b = (b_e)_{e \in E(\mathbb{Z}^d)} \in A$ where b is defined by $b_f = \begin{cases} a_f \text{ if } f \neq e \\ 1 \text{ if } f = e \end{cases}$.

Example:

- Therefore if $a = (0,0,0,1,0,\ldots)$ then $b = (0,0,1,1,0,\ldots)$ (where we change a_3 into a 1) satisfy $b \in A$.
- percolation $0 \leftrightsquigarrow \infty$ and e open.
- Nonexample: $\exists$ exactly 2 infinity clusters. If those 2 clusters are just 1 closed path from eachother away. Therefore if you turn that path one, you will get 1 cluster, so not 2 infinity clusters.

$A \subseteq \Omega$ is an DOWN-SET (DECREASING) EVENT if A^c is up-set.

STANDARD INCREASING COUPLING IID $U_e : e \in E, U_e \stackrel{d}{=} \text{Unif}[0,1]$.
$G_p :=$ graph with vertex set $\mathbb{R}^d$ edge set $\{e : U_e \leq p\}$. Behaves just like original model by parameter p.
Observe $p \leq q \Rightarrow G_p \subseteq G_q$.

Coupling of RVS X, Y is joint probability space for (X', Y') s.t. $X \stackrel{d}{=} X', Y \stackrel{d}{=} Y'$.

Lemma:
E is up-set, $p \mapsto \mathbb{P}_p(E)$ is non-decreasing.
Proof: $\mathbb{P}_p(E) = \mathbb{P}(G_p \text{ has } E) \leq \mathbb{P}(G_q \text{ has } E) = \mathbb{P}_q(E)$ where $p < q$.
G_p has E means that if E is for example the event that 2 points are connected, then that path between the two points, must be a path in G_p.
We use that E is up-set by the $\leq$ sign. The probability that event E first happens is $\mathbb{P}(G_p \text{ has } E)$. But if $p < q$ so we increase the probability there is even a higher probability that this event happens (since changing 0 to 1, is also possible, since up-set, which will also be an element of E, so higher change that we will have G_q has E).
Corollary:
$p \mapsto \theta(p), p \mapsto P_p(\text{percolation})$ are non-decreasing.
Proof: This is an immediate result since percolation is an up-set event.
Theorem:
$p \mapsto \theta(p)$ strictly increasing on $(p_c, 1)$
Proof: Tutorial exercise.
Theorem:
$\forall p : \mathbb{P}_p(\text{perc}) \in \{0, 1\}$.
Proof: Herefore we can use Kolmogorov's zero-one law: A sequence of independent (not necessarily identical distributed) random variables, adn E is a tail event, then $\mathbb{P}(E) \in \{0, 1\}$. Tail event is an event that is invariant under changing the values of finitely many variables. We see that our event is indeed a tail event, since

- if we have an infinite open path somewhere, and change some finite number of edges from open to close or visa versa, then this can break the infinite original path into finite paths. But there will be at least 1 infinite path.
- If we have e a finite path, and open some edges from open to close or vica versa, that will not produce an infinite path.

Hence percolation is invariant under changing the values of finitely many variables hence a tail event.
To proof Kolmogorov's zero-one law relies on to heavy material.

Lecture 2

Lemma:
$\forall \varepsilon > 0, \exists$ event F that only depends on status of finitely many edges (all edges of Λ_n,for some n) s.t. $\mathbb{P}(E\Delta F) < \varepsilon$.
Note: $A\Delta B = A \setminus B \cup B \setminus A$.
Proof:
$E_k := \{\Lambda_k \leftrightsquigarrow \infty\}$. Note that $E = \bigcup_k E_k$ (we can prove this by subsets, and using that $E_1 \subseteq E_2 \subseteq \ldots$. Therefore we see that$\exists k : |\mathbb{P}(E) - \mathbb{P}(E_k)| < \varepsilon/2$.
$E_{m,k} = \{\Lambda_k \leftrightsquigarrow \Lambda_m^c\}$ where $k < m$. (so we have a connection between Λ_k,and a point outside Λ_m). observe that this depends on a finite number of edges (namely Λ_k). Therefore $E_k = \bigcap_{m>k} E_{k,m}$,Exercise Tutorial. Furthermore $E_{k,m} \supseteq E_{k,m+1} \supset \ldots$. Hence we get $\mathbb{P}(E_k) = \lim_{m\to\infty} \mathbb{P}(E_{k,m})$. So $\exists m : |\mathbb{P}(E_k) - \mathbb{P}(E_{k,m})| < \varepsilon/2$
Note that $\mathbb{P}(E\Delta E_k) = |\mathbb{P}(E) - \mathbb{P}(E_k)| \leq \epsilon/2$ and $\mathbb{P}(E_k\Delta E_{k,m}) = |\mathbb{P}(E_k) - \mathbb{P}(E_{k,m})| \leq \epsilon/2$. Therefore $\mathbb{P}(E\Delta E_{k,m}) \leq \mathbb{P}(E\Delta E_k) + \mathbb{P}(E_k\Delta E_{k,m})$. This is because $E\Delta E_{k,m} \subseteq E\Delta E_k \cup E_k\Delta E_{k,m}$.
Hence $\mathbb{P}(E\Delta E_{k,m}) < \varepsilon$ so proven by chosing $F = E_{k,m}$ for certain k, m.
LIMITS CAN WORK SINCE UNION AND INTERSECTION, SEE PROBABILITY THEORY

$T : \mathbb{Z}^d \to \mathbb{Z}^d$ is TRANSLATION if $T(z) = z + c$ for $c \in \mathbb{Z}$.
(So $(z_1, \ldots, z_d) \mapsto (z_1 + c_1, \ldots, z_d + c_d)$.
Example: $e \in E(\mathbb{Z}^d)$ corresponds to $T(e)$. So if $e = uv$ then $T(e) = T(u)T(v)$.
Example: Event $A \subseteq \{0,1\}^{E(\mathbb{Z}^d)}$ corresponds to $T(A) := \{T \in A\}$ s.t. when $(X_e)_{e\in E}$ corresponds to original edge status, then $Y_e = X_{T^{-1}}(e)$. So if A is the event that e is open, where e is the edge between $(0,0)$ and $(1,0)$ and $T(z) = z + (1,1)$ then $T(A) = \{T^{-1}(e)$ open$\}$,so the event that the edge between $\{-1,-1)$ and $(0,-1)$ is open.
Note: $\mathbb{P}(T(A)) = \mathbb{P}(A)$.
A TRANSLATION INVERSE if $A = T(A)$ for all translations T.
Example: $\{$all edge open$\}$, $\{$percolation$\}$.
Non-example: $\{e$ is open$\}$, $\{0 \leftrightsquigarrow \infty\}$.

$E = \{$percolation$\}, F = \{$as in lemma$\}$. F depends on Δ_m. Take $T : z \mapsto z + (1000m, 0, \ldots, 0)$. $F, T(F)$ are independent. $\mathbb{P}(F \cap T(F)) = \mathbb{P}(F)\mathbb{P}(T(F)) = \mathbb{P}(F)^2$.
$E\Delta(F\cap T(F)) \subseteq E\Delta F \cup E\Delta T(F) = E\Delta F \cup T(E)\Delta T(F)$. Note that $T(E)\cap T(F) = T(E\Delta F)$.
Therefore $\mathbb{P}(E\Delta(T \cap T(F)) \leq 2\mathbb{P}(E\Delta F) < 2\epsilon$.
Therefore $|\mathbb{P}(E)(E) - \mathbb{P}(F)^2| = |\mathbb{P}(E) - \mathbb{P}(F \cap T(F))| \leq \mathbb{P}(E\Delta(F \cap T(F)) \leq 2\epsilon$.

$|\mathbb{P}(E)^2 - \mathbb{P}(F)^2| = |(\mathbb{P}(E) - \mathbb{P}(F))(\mathbb{P}(E) + \mathbb{P}(F))| \leq 2|\mathbb{P}(E) - \mathbb{P}(F)| < 2\epsilon$.
Therefore $\mathbb{P}(E) - \mathbb{P}(E)^2| \leq |\mathbb{P}(E) - \mathbb{P}(F)^2| + |\mathbb{P}(F)^2 - \mathbb{P}(E)^2| < 4\epsilon$.

Since $\epsilon to 0$, we have in fact $\mathbb{P}(E) = \mathbb{P}(E)^2$.

Theorem:
$E = \{\text{percolation}\}$ Then $\mathbb{P}_p(E) \in \{0, 1\}$.
Proof:
By above reasoning, we see that for $E = \{\text{percolation}\}$ we have $\mathbb{P}(E) = \mathbb{P}(E)^2$ which is only possible if $\mathbb{P}(E) \in \{0, 1\}$.

$p_c = \inf\{p : \mathbb{P}_p(\text{percolation}) > 0\} = \inf\{p : \mathbb{P}_p(\text{percolation}) = 1\}$, furthermore $p_c = \sup\{p : \mathbb{P}_p(\text{percolation}) = 0\} = \sup\{p : \theta(p) = 0\} = \inf\{p : \theta(p) > 0\}$
For $0 \leq p < p_c$ we see that $\mathbb{P}_p(\text{percolation}) = 0$. For $p_c < p \leq 1$ we see that $\mathbb{P}_p(\text{percolation}) = 1$. For $p = p_c$ it is still unknown what happends. Observe discontinuous.

Now we can ask ourself is $\theta(p)$ continuous? Note that if $\mathbb{P}_{p_c}(\text{percolation}) = 1 \Rightarrow \theta(p_c) > 0 \Rightarrow$ discontinuous (since for $0 \leq p < p_c$ we see that $\theta(p) = 0$.
Theorem:
$p \mapsto \theta(p)$ is continuous from the right in $d > 1$,i.e., $\lim_{q \searrow p} \theta(q) = \theta(p), \forall p \in (0, 1)$.
Proof:
Consider standard increasing coupling. $E_q = \{\text{in } G_q, 0 \leftrightsquigarrow \infty\}$
Claim:
$E_p = \bigcap_{q>p} E_q$
Proof of claim:

- $\subseteq$ obvious since if $q > p$ then $G_q \supseteq G_p$.
- $\subseteq$ Exercise $P_1, P_2, \ldots$ infinite paths starting at origin in $\mathbb{Z}^d$, $\exists \infty$ path P in $\mathbb{Z}^d$ s.t. $\forall e \in E(p), \exists n_1 < n_2 < \ldots$ with $e \in E(P_{n_i}), \forall i$.
  Want to show $E_p \supseteq \bigcap_{q>p} E_q$. Fix $q_1, q_2, \ldots > p$ s.t. $q_i \to p$. Suppose $\bigcap_{i=1}^{\infty} E_{q_i}$ holds, then $\exists$ infinite paths Pi in G_{q_i} starting at origin.
  Exercise: $\exists \infty$ path P starting at origin s.t. each edge in ∞ many G_{q_i}.
  Fix $e \in E(P), n_1, n_2, \ldots ; e \in G_{q_{n_i}}, \forall i$, i.e. $U_e \leq q_{n_i}, \forall i$. so $U_e \leq \lim_{i \infty} q_{n_i} = p$ so $e \in G_p, \forall e \in E(p)$. Therefore P is in G_p.

Hence $\theta(p) = \mathbb{P}(E_p) = \mathbb{P}(\bigcap_{q>p} E_q) = \lim_{q \searrow p} \mathbb{P}(E_q) = \theta(q)$

Theorem:
in $d = 2$ then $\frac{1}{3} \leq p_c \leq \frac{2}{3}$.
Proof:
For $\frac{1}{3} \leq P_c$ enough to show $g(p) = 0, \forall p < \frac{1}{3}$.
$\{\text{origin} \leftrightsquigarrow \infty\} \overset{\forall n}{\subseteq} \{\text{origin} \leftrightsquigarrow \Lambda_n^c\} \subseteq \{\exists \text{path length } n \text{ starting at origin}\}$.
$\mathcal{P}_n = \{\text{all paths starting at origin, length } n\}$. Therefore

$$\begin{aligned}\theta(p) &\leq \sum_{p \in \mathcal{P}_n} \mathbb{P}_p\{\text{all edges of } p \text{ open}\} \\ &= \sum_{p \in \mathcal{P}_n} p^n = |\mathcal{P}_n| \cdot p^n\end{aligned}$$

Observe that $|\mathcal{P}_n| \leq 4 \cdot 3^{n-1}$. This is because from the origin you have 4 options. After that, you have 3 options left since you can not go back (since otherwise not a path). Hence we see that

$$\theta(p) \leq 4 \cdot 3^{n-1} \cdot p^n = \frac{4}{3}(3p)^n \xrightarrow{n \to \infty} 0$$

For $p_c \leq \frac{2}{3}$ we want to show that $\theta(p) > 0, \forall p > \frac{2}{3}$.
If cluster of origin is finite, then it has a contour, notation C(or.). Goes from inside a square to adjecent square crossing only closed edges, returning to starting point encloses origin.
$1 - \theta(p) = \mathbb{P}_p(\text{origin} \not\leftrightsquigarrow \infty) \leq \sum_{\text{contours enclosing origin}} \mathbb{P}(\text{all edges of} C \text{ closed})$
$= \sum_{n \geq 4} \sum_{C \in C_n} |C_n|(1-p)^n = \sum_{n \geq 4} n \cdot 3^{n-1}(1-p)^n$. Therefore we see that $1 - \theta(p) < \infty$ iff $p > \frac{2}{3}$. Therefore if $p < 1$ then $1 - \theta < 1$ hence $\theta > 0$.

Observe that if $p \approx \frac{2}{3}$ then the sum can be way larger. Therefore we have to restrict it to a specific length:
Fix n_0 s.t. $\sum_{n \geq n_0} n3^{n-1}(1-p)^n < \frac{1}{2}$. $e_1, \ldots, e_{n_0} \in E(\mathbb{Z}^d)$ are first n_0 edges on positive $x-$ axis with $e_i = (i-1, 0), (i, 0)$.
Let $A := \{0 \not\leftrightsquigarrow \infty\} \cap \{e_1, \ldots, e_{n_0}, \text{open}\}$ and
$\mathcal{D}_n :=$ contours that enclose $(0,0), (1,0), \ldots, (n_0, 0)$ of length n
$\mathbb{P}(A) \leq \sum_n |\mathcal{D}_{n > n_0}| \cdot (1-p)^n p^{n_0} \leq p^{n_0} \sum_{n > n_0} n3^{n-1}(1-p)^n < p^{n_0} \cdot \frac{1}{2}$.

$B := \{0 \leftrightsquigarrow \infty\} \cap \{e_1, \ldots, e_{n_0} \text{ open}\}$.
Therefore $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(e_1, \ldots, e_n \text{ open}) = p^{n_0}$.
Therefore $\mathbb{P}(B) = p^{n_0} - p(A) > p^{n_0}/2 > 0$.
MISSED

Lecture 3

Corollary:
$p_c \leq \frac{2}{3}, \forall d \geq 2$.

Dual graph

We proved that $p_c \geq \frac{1}{2d-1}$ in the lecture. In fact $p_c = (1+o_d(1)) \times \frac{1}{2d}$. Here $o_d(1)$ means that $\forall \varepsilon, \exists d_0; p_c \in \left(\frac{1-\varepsilon}{2d}, \frac{1+\varepsilon}{2d}\right)$

Planar graph graph $G = (V, E)$ that can be drawn in $\mathbb{R}^2$.
Plane graph planar and fixed drawing.
Planar graph has faces:=connected regions of $\mathbb{R}^k \setminus$ drawing
Dual graph place vertex inside each face of plane graph, connect these iff faces meet in an edge.
Notation: $G^* =$ Dual of G.
Example: (So in the integer grid, you get for example a shifted graph in fact). For each $e \in E(\mathbb{Z}^2), \exists! e^* \in E((\mathbb{Z}^2)^\star)$ that intersects it.

Define coupling between percolation on $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$ by e^* open iff e closed. Therefore $\mathbb{P}(e^* \text{ open}) = 1 - \mathbb{P}(e \text{ open}) = 1 - p$.

Crossing probability/event

rectangle:$R = \{a, \ldots, b\} \times \{c, \ldots, d\}$.
Horizontal crossing: $H(R) = \{\{a\} \times \{c, d\} \overset{R}{\leftrightsquigarrow} \{b\} \times \{c, d\}\}$
Vertical crossing: $V(R) = \{\{a, b\} \times \{c\} \overset{R}{\leftrightsquigarrow} \{a, b\} \times \{d\}\}$.

R is an $(n+1) \times n$ rectangle, say $\{0, \ldots, n\} \times \{1, \ldots, n\}$, then R^* is R but rotated.
Exercise: $\mathbb{P}(H(R)) + \mathbb{P}(H(R^*)) = 1$

If $p = \frac{1}{2}$ then $2\mathbb{P}(H(R)) = 1$, this is true $\forall n$.
Note that $\mathbb{P}_p(H(\tilde{R})) \geq \frac{1}{2}$ where $\tilde{R}$ is a square (so $n \times n$) (Let's denote this by $\boxminus$)

Harris'Lemma

$H(R), V(R)$ eample of up-sets depending only on finite edges.

Lemma:

$A, B \subseteq \{0,1\}^n$ up-sets ($n \in \mathbb{N}, p \in [0,1]$),then $\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A) \cdot \mathbb{P}_p(B)$.
Note: $\mathbb{P}_p(A) = \mathbb{P}_p((X_1, \dots, X_n) \in A)$ where X_i i.i.d. Be(p).
Note: Also FKG inequality
Proof:
$n = 1$, then A, B are $\emptyset, \{1\}, \{0,1\}$. We see that this is in fact trivial.
IH: assume true for $n-1$. Fix $A, B \subseteq \{0,1\}^n$.
Let $A_i = \{(x_1, \dots, x_{n-1}) \in \{0,1\}^{n-1} | (x_1, \dots, x_{n-1}, i) \in A\}$ with $i \in \{0,1\}$ Define B_i similair.
Then A_1, B_1 are upsets. Similair $A_0 \subseteq A_1, B_0 \subseteq B_1$. Therefore

$$\begin{aligned}
(\mathbb{P}(A_1) - \mathbb{P}(A_0))(\mathbb{P}(B_1) - \mathbb{P}(B_0)) &\geq 0 \\
\mathbb{P}(A_0)\mathbb{P}(B_0) + \mathbb{P}(A_1)\mathbb{P}(B_1) &\geq \mathbb{P}(A_0)\mathbb{P}(B_1) + \mathbb{P}(A_1)\mathbb{P}P(B_0) \\
(1-p)\mathbb{P}(A_0) + p\mathbb{P}(A_1) &= \mathbb{P}(A) \\
(1-p)\mathbb{P}(B_0) + p\mathbb{P}(B_1) &= \mathbb{P}(B) \\
(1-p)\mathbb{P}(A_0 \cap B_0) + p\mathbb{P}(A_1 \cap B_1) &= \mathbb{P}(A \cap B)
\end{aligned}$$

Therefore by induction step, we see that

$$\begin{aligned}
\mathbb{P}(A \cap B) &\geq (1-p)\mathbb{P}(A_0)\mathbb{P}(B_0) + p\mathbb{P}(A_1)\mathbb{P}(B_1) \\
&= (1-p)^2\mathbb{P}(A_0)\mathbb{P}(B_0) + (1-p)p\mathbb{P}(A_0)\mathbb{P}(B_0) \\
&+ p^2p\mathbb{P}(A_1)\mathbb{P}(B_1) + (1-p)p\mathbb{P}(A_1)\mathbb{P}(B_1) \\
&\geq (1-p)^2\mathbb{P}(A_0)\mathbb{P}(B_0) + p^2\mathbb{P}(A_1)\mathbb{P}(B_1) \\
&+ (1-p)p(\mathbb{P}(A_0)\mathbb{P}(B_1) + \mathbb{P}(A_1)\mathbb{P}(B_0)) \\
&= ((1-p)\mathbb{P}(A_0) + p\mathbb{P}(A_1))((1-p)\mathbb{P}(B_0) + p\mathbb{P}(B_1)) \\
&= \mathbb{P}(A)\mathbb{P}(B)
\end{aligned}$$

Due to this lemma, we see that $\mathbb{P}(H(R) \cap V(R^*)) \geq \mathbb{P}(H(R))\mathbb{P}(V(R^*))$.

A upset,B downset, then $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\mathbb{P}(B)$.

MAKE SOME SENSE FROM THIS BELOW

Russo-Seymour-Welsh theorem

$\forall \alpha > 0, \exists c(\alpha) > 0; \mathbb{P}_{\frac{1}{2}}(H(R)) \geq c(\alpha), \forall n, \forall \lceil n \cdot \alpha \rceil \times n$ rectangle R.
Proof:
Enough to show for your favourite $\alpha > 1$. Since if true for α then also true for $0 < \beta < \alpha$. (take for example $c(\beta) = c(\alpha)$).
For $\alpha \leq 1$, take $c(\alpha) = \frac{1}{2}$.
If $V(R)$ holds, $\exists$ a left most vertical crossing. Since R finite rectangle, then finitely many possibilities hence there must be at least a left one.

SEE PLOT NOTES

$$\mathbb{P}(H(R_1 \cup R_2)) \geq \mathbb{P}(H(R_1) \cap H(R_2) \cap V(R_1 \cap R_2))$$
$$c(\alpha)c(\alpha) \cdot \frac{1}{2}$$

Exercise: Fix R conditional on $V(R)$,let Γ be left most vertical crossing. Γ is undefined if $V(R)$ does not hold. Show that $\{\Gamma = \gamma\}$ depends only on edges to the left of γ, for all fixed vertical crossing of R.

Lecture 4

Some proof

FOTO 1
$\alpha = \frac{3}{2}$. With Q a $n \times n$ square, R a $2n \times 2n$ square s.t. $Q \subsetneq R$.
$E = V(Q) \cap \{\exists$path inside R from a vertical crossing in Q to right boundary of $R\}$
FOTO 2
Γ is left most vertical crossing of Q. γ vertical crossing of Q,and γ^* mirror image reflected in the line with arrow. Fix γ (non-random). $E_\gamma = \exists$ path from γ to the right side of R.
FOTO 3
Hence $\mathbb{P}_{\frac{1}{2}}(E_\gamma \cup E_{\gamma^*}) \geq \mathbb{P}_{\frac{1}{2}}(H(R)) \geq \frac{1}{2}$. Therefore $\mathbb{P}(E_\gamma) + \mathbb{P}(E_{\gamma^*}) \geq \mathbb{P}(E_\gamma \cup E_{\gamma^*})$. By symmetry, $\mathbb{P}(E_\gamma) + \mathbb{P}(E_{\gamma^*}) = 2\mathbb{P}(E_\gamma)$ hence $\mathbb{P}(E_\gamma) \geq \frac{1}{4}$

$E'_\gamma = \{$Right of R connected to γ with path not corssing $\gamma^*\}$

$$\mathbb{P}(E) \geq \sum_\gamma \mathbb{P}(E'_\gamma \cap \Gamma = \gamma) = \sum_\gamma \mathbb{P}(E'_\gamma | \Gamma = \gamma)\mathbb{P}(\Gamma = \gamma)$$

Note that $E'_\gamma, \Gamma = \gamma$ depends on disjoint sets of edges. Therefore

$$E'_\gamma \geq \frac{1}{4} \sum_\gamma \mathbb{P}(\Gamma = \gamma) = \mathbb{P}(V(Q)) \geq \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

Now take F to be a rectangle s.t. 2 squares R, R' overlap.
$F = H(Q) \cap E \cap \{E$ holds for $R'\}$. By Harris Lemma, we see that $\mathbb{P}(F) \geq \mathbb{P}(H(Q))\mathbb{P}(E)^2 \geq \frac{1}{2} \cdot \left(\frac{1}{8}\right)^2 = \frac{1}{128}$. This is for even n. For odd n, easy to reduce $\mathbb{P}(H(R)) \geq c(\alpha)$ for some $C(\alpha) > 0, \forall \lceil \frac{3}{2}n \rceil \times n$ rectangle,by looking at $n-1$, do the same and then taking the third power of $\frac{1}{128}$.

Harris' Theorem

In $d = 2$ we have $\theta(1/2) = 0$.
Corollary:
$p_c \geq \frac{1}{2}$.
Zhang's argument
Argument also true in dual $(p = \frac{1}{2})$
$\mathbb{P}(..) \geq c(3)^4 > 0$
Proof:
$A_1, A_2, \ldots$ ested sequences ... of ... size. $A_1, \ldots, A_i$ inside inner region of A_{i+1}.
Each A_i made of $4\, 3n_i \times n_i$ squares as before. Therefore $\mathbb{P}(0 \leftrightsquigarrow \infty) \leq \mathbb{P}(\text{every } A_i \text{ does not have} \ldots) = \mathbb{P}\left(\bigcap_i A_i \text{ does not...}\right) = \prod_i (A_i \text{ does not}) \leq \prod_{i=1}^{\infty} (1 - c(3)^4) = 0$
Theorem:
In $d = 2, \forall \in [0, 1], \mathbb{P}_p(\exists \geq 2 \text{ distinct } \infty \text{ clusters}) = 0$.
Proof:
$p \leq \frac{1}{2}$ then we are done, since no clusters since $p_c = \frac{1}{2}$.
Fix $p \geq \frac{1}{2}$ and z_1, z_2 s.t. $F_{z_1,z_2} = \{z_1 \leftrightsquigarrow \infty\} \cap \{z_2 \leftrightsquigarrow \infty\} \cap \{z_1 \not\leftrightsquigarrow z_2\}$. Start zhang from a larg size s.t. z_1, z_2 inside first annulus.
Therefore by Shang's argument $\mathbb{P}_{1/2}(\exists \text{ certain open surrounding } z_1 \& z_2) = 1$.
Therefore we see that $\mathbb{P}_p(\exists \text{ certain open surrounding } z_1 \& z_2) = 1$.
Therefore $\mathbb{P}(F) \leq \mathbb{P}(\text{no op en circuit surrounding } z_1, z_2 \text{ exists}) = 0$

$$\mathbb{P}(\exists \geq 2 \text{ distinct } \infty \text{ components}) = \mathbb{P}\left(\bigcup_{z_1 \neq z_2} F_{z_1,z_2}\right) < \sum_{z_1,z_2} \mathbb{P}(F_{z_1,z_2}) = 0$$

A PERCOLATION MODEL $\approx$ A random subset of $E(\mathbb{Z}^d)$.
1 INDEPENDENT PERCOLATION: if $\{e_1, \text{open}\}, \ldots, \{e_k, \text{open}\}$ are independent,$\forall e_1, \ldots, e_k (\forall k)$ that do not share endpoints.
Example:
$\forall v \in \mathbb{Z}^d$, is red with probability r and blue with $1 - r$. Open edges="monochromatic" ones (monochromatic is 1 colour). So $\mathbb{P}(e \text{ open}) = r^2 + (1 - r)^2 = p$. Edges not independent. We see this by taking a 4 cycle. If the edges e_1, e_2, e_3 are open then e_4 is also open. So $\mathbb{P}(e_4 \text{ open} | e_1, e_2, e_3 \text{ open}) = 1 \neq p$ unless $r = 1$ or $r = 0$. But we see that this is 1-independent. Each edge depends on 2 coins if $e_1, \ldots, e_k$ do not share endpoints, sets of coins disjoint.

Lemma:
$\exists p_1 < 1$ s.t. in any 1-independent percolation model on $\mathbb{Z}^2$, if $\mathbb{P}(e \text{ open}) \geq p_1, \forall e \in E(\mathbb{Z}^2)$, then $\mathbb{P}(\exists \infty \text{ cluster}) > 0$.
Proof:

$$\begin{aligned} \mathbb{P}(0 \not\leftrightsquigarrow \infty) &\leq \sum_{C \text{ circuit surrounding } e} \mathbb{P}(\text{ all edges of } C \text{ closed}) =\leq \sum_{C} (1-p_1)^{E(C)/2} \\ &\leq \sum_{n\geq 1} n3^{n-1}(\sqrt{1-p_1})^n \\ &< 1 \text{ iff } p_1 \text{ close to } 1 \\ \mathbb{P}(0 \leftrightsquigarrow \infty) &> 0 \end{aligned}$$

Note that $\sum\limits_{n\geq 0} x^n = \frac{1}{1-x}$ for $-1 < x < 1$. Hence $\sum\limits_{n\geq 1} nx^{n-1} = \left(\frac{1}{1-x}\right)^2$
Lemma:
$\exists P2 < 1$ s.t. the $p \in [0,1], \in \mathbb{R}$ are s.t. $\mathbb{P}_p(\boxminus) \geq p_2$ then $\theta(p) > 0$, with $\boxminus$ of size $3n \times n$.
Proof:
Make a coupling with a 1-independent percolation model $e = (i,j)(i+1,j)$. Let

$$R_e = \{(2i-n)(n+1), \ldots, 2(i+1)n\} \times \{(2j-1)n+1, \ldots, 2nj\}$$

e open if for i horizontal $H(R) \cap V(\ldots)$. Similarly if e vertical, $\mathbb{P}(e \text{ open}) \geq p^2$. Assume $*p_2$ suff. large to be discussed. By definition of e open i if $\exists\infty$ path in 1-indepedent model $\Rightarrow \exists\infty$ path in original.
$\mathbb{P}_p(\exists\infty \text{ cluster in original}) \geq \mathbb{P}(\exists\infty \text{ clusters in 1-indep.}) > 0$ where last if $p_2^2 \geq p_1$ (in previous lemma). So can take $p_2 = \sqrt{p_1} < 1$.

Lecture 5

We want to show that $p_c = \frac{1}{2}$ in $d = 2$. By Harris's theorem. For $p_c \leq \frac{1}{2}$ enough. That is $\forall p > \frac{1}{2}$ we have $\mathbb{P}_p[H(1, \ldots, 3n\} \times \{1, \ldots, n\})] \xrightarrow{n \to \infty} 1$.
$\{e$ pivotal for $H(R)\} =$ changing status of (only) e changes whether $H(R)$ occurs.
Exercise:
If e is pivotal, then $\exists$ paths from end pointsof $e^\star$ (open in dual) to top and bottom of R.

$A \subseteq \{0,1\}^n$. Consider $\mathbb{P}_p(A) = \mathbb{P}(X_1, \ldots, X_n) \in A)$ with $X_1, \ldots, X_n$ iid $\text{Be}(p)$. Coordinate i pivotal for A if $(X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n) \in A)$ and
$(X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n) \notin A$ (or vica versa)
$\text{Inf}_i(A) = \mathbb{P}(i$ is pivital) here Inf is influence.

Theorem: Margulis-Russo Formula
$A \subseteq \{0,1\}^n$ up-set. Then

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{i=1}^{n} \text{Inf}_i(A)$$

Proof:
Consider situation where $X_i \sim \text{Be}(p_i)$ (still independent). Therefore

$$\mathbb{P}_{p_1,\ldots,p_n}(A) = \sum_{x \in A} \prod_{j=1}^{n} p_i^{x_j}(1-p_j)^{1-x_j}$$

Note for $n < \infty$, we see polynomial in $p_1, \ldots, p_n$. Therefore $\frac{\partial}{\partial p_i}\mathbb{P}(A)$ exists. Hence

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\mathbb{P}_{p_1,\ldots,p_n}(A)\Big|_{p_1=\ldots=p_n=p}$$

Therefore

$$\mathbb{P}_{p_1,\ldots,p_n}(A) = \sum_{\substack{(x_1,\ldots,x_{i-1},x_{i+1},x_n)\in\{0,1\} \\ (x_1,\ldots,x_{i-1},0,x_{i+1},x_n)\in A \\ (x_1,\ldots,x_{i-1},1,x_{i+1},x_n)\in A}} \prod_{j\neq i} p_j^{x_j}(1-p_j)^{1-x_j} + \sum_{\substack{(x_1,\ldots,x_{i-1},x_{i+1},x_n)\in\{0,1\} \\ (x_1,\ldots,x_{i-1},0,x_{i+1},x_n)\notin A \\ (x_1,\ldots,x_{i-1},1,x_{i+1},x_n)\in A}} p_i \prod_{j\neq i} p_j^{x_j}(1-p_j)^{1-x_j}$$

We see that this is an equality, since A is an upset. Hence

$$\frac{\partial}{\partial p_i}\mathbb{P}_{p_1,\ldots,p_n}(A) = \sum_{\substack{(x_1,\ldots,x_{i-1},x_{i+1},x_n)\in\{0,1\} \\ (x_1,\ldots,x_{i-1},0,x_{i+1},x_n)\in A \\ (x_1,\ldots,x_{i-1},1,x_{i+1},x_n)\in A}} \prod_{j\neq i} p_j^{x_j}(1-p_j)^{1-x_j}$$

$$= \mathbb{P}(i \text{ is pivotal (A IS UP-SET)} = \text{Inf}_i(A)$$

Proposition:
$\forall p > \frac{1}{2}, \lim_{n\to\infty} \mathbb{P}_p(H(\{1,\ldots,3n\}\times\{1,\ldots,n\})) = 1.$
Proof:
From tutorial exercise $\mathbb{P}_{1/2}(\underline{0} \leftrightsquigarrow \Lambda_n^c) \leq n^{-c}$ for some $c > 0$.
$\epsilon > 0, \frac{1}{2} < p < 1$. Pick n large tbd

$$\Psi(q) := \mathbb{P}_q(H(\{1,\ldots,3n\}\times\{1,\ldots,n\})) \text{ with } q \in [0,1]$$

Assume $\Psi(p) < 1-\epsilon$. We know $\forall q \geq \frac{1}{2}$ that $\Psi(q) \geq c(3) > \epsilon$ therefore $\epsilon < \Psi(q) < 1-\epsilon$, forall $\frac{1}{2} \leq q \leq p$.

$$\begin{aligned}\Psi'(q) &\geq \sum \mathrm{Inf}_i \geq c\cdot\Psi(q)(1-\Psi(q)) \times \ln\left(\frac{1}{\max_e \mathbb{P}_q(e \text{ pivotal})}\right)\\ &> c\cdot\epsilon(1-\epsilon)\ln\left(\frac{1}{\max_e \mathbb{P}_q(e \text{ pivotal})}\right)\end{aligned}$$

If e ivotal, then top and bottom each have path in dual to endpoints of e^*. We call these end points u^* and $v^\star$. Therefore

$$\begin{aligned}\mathbb{P}(e \text{ pivotal}) &\leq \mathbb{P}(u^* \leftrightsquigarrow u^* + \Lambda^c_{\frac{n}{1000}} \text{ in dual}) + \mathbb{P}(v^* \leftrightsquigarrow v^* + \Lambda^c_{\frac{n}{1000}} \text{ in dual})\\ &= 2\mathbb{P}_{1-q}\left(\underline{0} \leftrightsquigarrow \Lambda^c_{\frac{n}{1000}}\right) \leq 2\mathbb{P}_{\frac{1}{2}}\left(\underline{0} \leftrightsquigarrow \Lambda^c_{\frac{n}{1000}}\right)\\ &\leq 2\cdot\left(\frac{n}{1000}\right)^{-c}\end{aligned}$$

Therefore

$$\Psi'(q) \geq c\cdot\epsilon\ln(1-\epsilon)\left(c\ln(n) + \text{constant}\right)$$

Therefore $\Psi'(q) \to \infty$ when n to infinity. WLOG, n s.t. $\Psi'(n) > k$ for k large to be discussed. Since this is independent of q we see that this holds $\forall\frac{1}{2} \leq q \leq p$.
Therefore

$$\Psi(p) = \Psi(\frac{1}{2}) + \int_{\frac{1}{2}}^{p} \Psi'(q)dq > K(p-\frac{1}{2}) > 1$$

where last choice of K. This can not be, since $\Psi(p)$,is a probability. Therefore $\Psi(p) < 1-\epsilon$ must be false.

Talagrand's inequality

$\exists$ universal constant s.t. $\forall n \in \mathbb{N}, \forall 0 \leq p \leq 1, \forall A \subseteq \{0,1\}^n$ we have

$$\sum_{i=1}^{n} \text{Inf}_i(A) \geq c\mathbb{P}(1-\mathbb{P}(A)) \times \ln\left(\frac{1}{\max\limits_i \text{Inf}_i(A)}\right)$$

Proof:
First show for $p = \frac{1}{2}$ deduce then $\forall p$. Note that $A \subseteq \{0,1\}^n$ corresponds to function $f : \{\pm 1\}^n \to \{\pm 1\}$. By setting

$$f(x_1, \dots, x_n) = \begin{cases} +1 \text{ if } \left(\frac{x_1+1}{2}, \dots, \frac{x_n+1}{2}\right) \in A \\ -1 \text{ otherwise} \end{cases}$$
$$V = \{f : \{\pm 1\}^n \to \mathbb{R}\} \text{ is a vector space,take as origin } f : \{\pm 1\}^n \to 0$$

Note that each vector has 2^n points in $\{\pm 1\}^n$. Store $f(x)$ for each of these. So $\dim(V) = 2^n$. Basis: $f_y(x) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$ with $y \in \{\pm 1\}^n$. Therefore $f_y(x) = \mathbf{1}_y$.
Pick inner product $\langle f, g \rangle := \mathbb{E} f(x)g(x)$ where $X = (x_1, \dots, x_n)$ iid $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$.
Let $X = (x_1, \dots, x_n) \in \mathbb{R}^n, S \subseteq [n]$, then $X^S = \prod\limits_{i \in S} x_i$. (So $x^{\{1,3\}} = x_1 \cdot x_3$. Furthermore $X^{\emptyset} = 1$.
Lemma:
$X^S; S \subseteq [n]$ form orthonormal basis for V. We have 2^n ... of ... pick $T, S \subseteq [n]$. Therefore

$$\langle x^S, x^T \rangle = \mathbb{E}\left(\prod_{i \in S \cap T} x_i^2 \prod_{i \in S \Delta T} X_i\right) = \mathbb{E} \prod_{i \in S \Delta T} X_i = \prod_{i \in S \Delta T} \mathbb{E}[X_i] = \begin{cases} 0 \text{ if } S \neq T \\ 1 \text{ if } S = T \end{cases}$$

Note that $\mathbb{E}[X_i] = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$.

Therefore $f \in V$ can be written as

$$f = \sum_{S \subseteq [n]} \hat{f}(s) \cdot x^S$$

We call this Fourier-Walsh coefficient (decoposition).
By Plancherel's identity, we see that

$$\langle f, g \rangle = \sum_{S \subseteq [n]} \hat{f}(s) \cdot \hat{g}(s)$$

Proof:

$$\begin{aligned}\langle f, g\rangle &= \left\langle \sum_{S\subseteq[n]} \hat{f}(s)\cdot x^S, \sum_{T\subseteq[n]} \hat{g}(T)\cdot x^T \right\rangle = \sum_S \sum_T \hat{f}(s)\hat{g}(t)\langle x^S, x^T\rangle \\ &= \sum_{S=T} \hat{f}(s)\hat{g}(t)\cdot 1\end{aligned}$$

By Parseval's identity $\langle f, f\rangle = \sum_S \hat{f}(s)^2 = 1$ if f is ± 1 valued.
Write $Ef = Ef(x_1, \dots, x_n)$ and $\text{var} f = \text{var}(f(x_1, \dots, x_n)) = Ef^2 - (Ef)^2$
$\mathbb{E}f = \sum_s \hat{f}(s)\mathbb{E}x^s = \hat{f}(\emptyset)$
$\text{Var}(f) = \langle f, f\rangle - \hat{f}(\emptyset)^2 = \sum_{S\neq\emptyset} \hat{f}(s)^2$. if f is ± 1 valued, then we have $\text{Var}(f) = 1 - \hat{f}(\emptyset)^2$.
So if f is ± 1 valued, then $\text{Var}(f) = 1 - (\mathbb{E}(f))^2 = 1 - (1\cdot\mathbb{P}(f = 1) - \mathbb{P}(f = -1))^2 = 4\mathbb{P}(f = 1) - 4\mathbb{P}(f = 1)^2 = 4\mathbb{P}(f = 1)(1 - \mathbb{P}(f = 1))$.
Compare this to talagrand.

Lecture 6

Operators

$V = \{f : \{\pm 1\}^n \to \mathbb{R}\}$ with $\langle f, g\rangle = \mathbb{E} f(X) g(X)$ where $X = (X_1, \dots, X_n)$ iid Unif($\{\pm 1\}$)¿
$\text{Inf}_i(f) := \mathbb{P}(f(X_1, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n) \neq f(X_1, \dots, X_{i-1}, -1, X_{i+1}, \dots, X_n))$
$x \in \{\pm 1\}^n$ then $x^{i \to j} = (X_1, \dots, X_{i-1}, y, X_{i+1}, \dots, n)$

$f = \sum_{S \subseteq [n]} \hat{f}(s) \cdot x^S$.
i th differential operation $f \mapsto D_i f$ with $(D_i f)(x) = \frac{1}{2}\left(f(x^{i\to 1}) - f(x^{i \to -1})\right)$
i th expectation operation $f \mapsto \mathbb{E}_i f$ with $(\mathbb{E}_i f)(x) = \frac{1}{2}\left(f(x^{i\to 1}) + f(x^{i \to -1})\right) = \mathbb{E} f(x_1, \dots, X_{i-1}, X, X_{i+}$
Unif($\{\pm 1\}$)
Exercise:
Show that $f = x_i D_i f + E_i f$.
Note $D_i, \mathbb{E}_i$ are linear operators, so $D_i(\lambda f + g\mu) = \lambda D_i(f) + \mu D_i(g)$, sim. for $\mathbb{E}_i$.
Lemma:
$D_i f = \sum_{S \ni i} \hat{f}(s) \cdot x^{S \setminus \{i\}} = \sum_{S \not\ni i} \hat{f}(s) x^S$
Proof:

$$D_i f = D_i \sum_S \hat{f}(s) x^S = \sum_S \hat{f}(s) D_i(x^S)$$

$$D_i x^S = \frac{1}{2}\left((x^S)^{i \to +1} - (x^S)^{i \to -1}\right) = \begin{cases} 0 \text{ if } i \notin S \\ x^{S\setminus\{i\}} \text{ if } i \in S \end{cases}$$

$$D_i f = \sum_{S \ni i} \hat{f} x^{S \setminus \{i\}}$$

$$E_i x^S = \frac{1}{2}\left((x^S)^{i\to 1} + (x^s)^{i \to -1}\right) = \begin{cases} 0 \text{ if } i \in S \\ x^S \text{ if } i \notin S \end{cases} \quad = \sum_{S \not\ni i} \hat{f}(s) x^S$$

Note: $f \to \{\pm 1\}$ then
$D_i f = \begin{cases} 0 \text{ if } f(x^{i \to +1}) = f(x^{i \to -1}) \\ \pm 1 \text{ otherwise} \end{cases}$. Therefore

$$\text{Inf}_i(f) = \mathbb{E}|D_i f| = \mathbb{E}(D_i f)^2 = \langle D_i f, D_i f\rangle = \sum_{S \ni i} \hat{f}(s)^2$$

$$\sum_{i=1}^n \text{Inf}_i(f) = \sum_i \sum_{s \ni i} \hat{f}(s)^2 = \sum_S \sum_{i \in S} \hat{f}(s)^2 = \sum_S |S| \hat{f}(s)^2$$

Example:
$f : \{\pm 1\}^n \to \{\pm 1\}$ is non-decreasing in all coordinates,then $\mathrm{Inf}_i(f) = \hat{f}(\{i\})$

$x \in \{\pm 1\}^n, -1 \leq \rho \leq 1$, then $Y = (Y_1, \dots, Y_n)$ is $\rho-$ CORRELATED TO X if Y_i independent, and $Y_i = \begin{cases} X_i \text{ with prop } \frac{1+\rho}{2} \\ -X_i \text{ with prop } \frac{1-\rho}{2} \end{cases}$ Notation: $Y \sim N_p(X)$.
Example:
If $\rho \geq 0$ then $Y_i = X_i$ with probability ,ρ and resampled ± 1 with probability $\frac{1}{2}$ otherwise.

NOISE OPERATOR $f \to T_\rho f$ given by $(T_\rho f)(x) = \mathbb{E}_{Y \sim N_\rho(x)} f(Y)$, note: linear
NOISE STABILITY $\mathrm{Stab}_\rho(f) = \langle f, T_\rho f \rangle = \mathbb{E}_x \mathbb{E}_{Y \sim N_\rho(X)} f(x) f(y)$
observation:
$\mathrm{Stab}_\rho(f) = \mathbb{P}(f(x) = f(y)) - \mathbb{P}(f(x) \neq f(y)) = 1 - 2\mathbb{P}(f(x) \neq f(y))$.
The more closer $\mathrm{Stab}_\rho(f)$ is to 1, the smaller the error.

Lemma:
$T_\rho f = \sum_S \rho^{|S|} \hat{f}(s) x^S$.
Proof:

$$T_\rho f = \sum_S \hat{f}(S) T_\rho x^S \tag{6.1}$$

$$T_\rho x^S = \mathbb{E}_{Y \sim N_\rho(x)} Y^S = \prod_{i \in S} \mathbb{E}_{Y \sim N_\rho(x)} Y_i$$

$$\mathbb{E}_{Y \sim N_\rho(x)} Y_i = \left(\frac{1+\rho}{2} x_i + \frac{1-\rho}{2} - x_i \right) = \rho x_i$$

$$T_\rho x^S = \prod_{i \in S} \rho x_i = \rho^{|s|} \prod_{i \in S} x_i$$

$$T_\rho f = \sum_S \hat{f}(s) \rho^{|s|} \prod_{i \in S} x_i = \sum_s \hat{f}(s) \rho^{|S|} x^S$$

Corollary:

$$\mathrm{Stab}_\rho(f) = \langle f, T_\rho f \rangle = \sum_S \rho^{|S|} \hat{f}(s)^2$$

$$\sum_i \mathrm{Stab}_\rho(D_i f) = \sum_s |S| \rho^{|S|-1} \hat{f}(s)$$

Exercise:
$\langle f, T_\rho g \rangle = \langle T_\rho f, g \rangle$ and $T_\rho(T_r(f)) = T_{\rho \circ r} f$. (we can do by Parseval and using (6.1).)

P-norm

P-NORM $\|f\|_p = \sqrt[p]{\mathbb{E}|f|^p}$ for $p > 0$.
We see that $\mathbb{E}^4 \geq (\mathbb{E}f^2)^2 \Rightarrow \|f\|_4 \geq \|f\|_2$.
T CONTRACTION if $\|Tf\|_p \leq \|f\|_p, \forall f$.

Theorem (2,4) Hupercontractivity:
$\|T_{1/\sqrt{3}}f\|_4 \leq \|f\|_2$ i.e. $\mathbb{E}\left(T_{\frac{1}{\sqrt{3}}}f\right)^4 \leq (\mathbb{E}f^2)^2$
Proof:
Induction on n. $f : \{\pm 1\}^n \to \mathbb{R}$. Base $n = 0$ so function does not depend on any constants, hence f is constant. Therefore $T_{\frac{1}{\sqrt{3}}}f = f$, therefore we see that we get equality

Now assume holds for any g that depends on less then n coordinates. So $\mathbb{E}\left(T_{\frac{1}{\sqrt{3}}}g\right)^4 \leq (\mathbb{E}g^2)^2$. Let $T = T_{\frac{1}{\sqrt{3}}}, d = D_n f, e = \mathbb{E}_n f, f = x_n d + e$ and $Tf = T(x_n \cdot d) + Te$. Therefore

$$
\begin{aligned}
Tf &= Tx_n \cdot Td + Te = \frac{1}{\sqrt{3}}x_n Td + Te \\
\mathbb{E}(Tf)^4 &= \left(\frac{1}{\sqrt{3}}\right)^4 \mathbb{E}\left(x_n^4\right)(Td)^4 + 4\left(\frac{1}{\sqrt{3}}\right)^3 \mathbb{E}x_n^3(Td)^3 Te \\
&\quad + 6\left(\frac{1}{\sqrt{3}}\right)^2 \mathbb{E}x_n^2(Td)^2(Te)^2 + 4\frac{1}{\sqrt{3}}\mathbb{E}x_n(Td)(Te)^3 + \mathbb{E}(Te)^4 \\
&= \left(\frac{1}{\sqrt{3}}\right)^4 \mathbb{E}Td^4 + 6\left(\frac{1}{\sqrt{3}}\right)^2 \mathbb{E}(Td)^2(Te)^2 + \mathbb{E}(Te)^4 \\
&\leq \mathbb{E}(Td)^4 + 2\mathbb{E}(Td)(Te)^2 + \mathbb{E}(Te)^4 \\
&\overset{CZ}{\leq} \mathbb{E}(Td)^4 + 2\sqrt{\mathbb{E}(Td)^4}\sqrt{\mathbb{E}(Te)^4} + \mathbb{E}(Te)^4 \\
&= \mathbb{E}(Td)^4 + 2\left(\sqrt[4]{\mathbb{E}(Td)^4}\right)^2\left(\sqrt[4]{\mathbb{E}(Te)^4}\right)^2 + \mathbb{E}(Te)^4 \\
&\overset{IH}{\leq} \mathbb{E}(Td)^4 + 2\left(\sqrt{\mathbb{E}d^2}\right)^2\left(\sqrt{\mathbb{E}e^2}\right)^2 + \mathbb{E}(Te)^4 \\
&= \mathbb{E}(Td)^4 + 2\mathbb{E}d^2\mathbb{E}e^2 + \mathbb{E}(Te)^4 \\
&\overset{IH}{\leq} (\mathbb{E}d^2)^2 + 2\mathbb{E}d^2\mathbb{E}e^2 + (\mathbb{E}e^2)^2 \\
&= \left(\mathbb{E}d^2 + \mathbb{E}e^2\right)^2 \qquad (6.2) \\
\mathbb{E}(f^2) &= \mathbb{E}\left((x_n d + e)^2\right) = \mathbb{E}(x_n^2 d^2 + 2x_n de + e^2) = \mathbb{E}(e^2) + \mathbb{E}(d^2)
\end{aligned}
$$

Note that from (6.2) we have $\mathbb{E}(Tf)^4 \leq \left(\mathbb{E}(d^2) + \mathbb{E}(e^2)\right)^2$, from last line $\mathbb{E}(d^2)+\mathbb{E}(e^2) = \mathbb{E}(f)^2$.

Cauchy Schwarz: $\langle f, g \rangle \leq \|f\|_2 \cdot \|f\|_2$.
Proof: Exercise

Hölder's inequality:
$\langle f, g \rangle = \|f\|_p \|g\|_q$ if $\frac{1}{p} + \frac{1}{q} = 1$,with $q, p > 0$.

Theorem (4/3,2) Hypercontractivity:
$\|T_{\frac{1}{\sqrt{3}}} f\|_2 \leq \|f\|_{\frac{4}{3}}$.
Proof:

$$\|T_{\frac{1}{\sqrt{3}}} f\|_2^2 = \left\langle T_{\frac{1}{\sqrt{3}}} f, T_{\frac{1}{\sqrt{3}}} f \right\rangle = \left\langle f, T_{\frac{1}{\sqrt{3}}} T_{\frac{1}{\sqrt{3}}} f \right\rangle \overset{\text{Holder}}{\leq} \|f\|_{4/3} \cdot \left\| T_{\frac{1}{\sqrt{3}}} T_{\frac{1}{\sqrt{3}}} f \right\|_4 \overset{2,4}{\leq} \|f\|_{4/3} \left\| T_{\frac{1}{\sqrt{3}}} f \right\|_2$$

Lecture 7

Corollary:
If $f : \{\pm 1\}^n \to \{-1, 0, 1\}$ then

$$\text{Stab}_{\frac{1}{3}} f = \langle f, T_{\frac{1}{3}} f \rangle \leq (\mathbb{E}(f))^{3/2}$$

Note that

$$\langle f, T_{\frac{1}{3}} f \rangle \overset{L7.1}{=} \left\langle f, T_{1/\sqrt{3}} \circ T_{1/\sqrt{3}} \circ f \right\rangle \overset{L7.2}{=} \langle T_{1/\sqrt{3}} f, T_{1/\sqrt{3}} f \rangle = \|T_{1/\sqrt{3}} f\|_2^2$$
$$\leq \|f\|_{4/3}^2 \left(\mathbb{E}|f|^{4/3}|\right)^{3/4 \cdot 2} = (\mathbb{E}(f))^{3/2}$$

L7.1 This follows from $T_\rho f = \sum \rho^{|S|} \hat{f}(s) x^S$
L7.2 follows from $\langle T_\rho f, g \rangle = \sum_S \hat{g}(s) \rho^{(s)} \hat{f}(s) = \langle f, T_\rho g \rangle$
After that we use the $(4/3, 2)$ hypercontracitivity, and $f \mapsto \{-1, 0, 1\}$.

Corollary:
$\forall f : \{\pm 1\}^n \to \{\pm 1\}$ then $\text{Stab}_{1/3}(D_i f) \leq \text{Inf}_i(f)^{3/2}$.
Proof:

$$(D_i f)(x) = \frac{1}{2} \left(f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, -1, x_{i+1}, \ldots, x_n)\right)$$

$\mathbb{E}|D_i f| = \text{Inf}_i f$

MISSED SOMETHING

Talagrand ($p = 0.5$ **case**
$\exists c > 0$ s.t. $\forall n, \forall f : \{\pm 1\}^n \to \{\pm 1\}$ then

$$\sum_i \text{Inf}_i(f) \geq c \cdot \text{var}(f) \cdot \ln \left(\frac{1}{\max_i \text{Inf}_i(f)} \right)$$

Proof:
$I := \sum_i \text{Inf}_i(f)$, Let $M > 0$ tbd.

$$\sum_{1 \leq |s| \leq M} \hat{f}(s)^2 \leq 3^M \sum_{|S| \geq 1} \left(\frac{1}{3}\right)^{|S|-1} |S| \hat{f}(S)^2 = 3^M \sum_i \text{Stab}_{\frac{1}{3}} (D_i f)$$

This last equality follows from

$$D_i f = \sum_{S \ni i} \hat{f}(S) x^{S \setminus \{i\}}$$
$$T_{1/3} f = \sum_{s \ni i} \left(\frac{1}{3}\right)^{|S|-1} \hat{f}(S) x^{S \setminus \{i\}}$$

Continuing we get

$$\sum_{1 \le |s| \le M} \hat{f}(s)^2 \le 3^M \sum_i \mathrm{Inf}_i(f)^{3/2} \le 3^M \cdot \sqrt{\max_i \mathrm{Inf}_i f} \cdot \sum_i \mathrm{Inf}_i(f) =^M \cdot \sqrt{\max_i \mathrm{Inf}_i f} \cdot I$$

Now for the other part

$$\sum_{|S|>m} \hat{f}(s)^2 \le \frac{I}{M} = \frac{1}{m} \sum_S |S| \hat{f}(s)^2$$

Now use that

$$\mathrm{var} f = \sum_{|S|>1} f\hat{(s)}^2 \overset{L7.3}{\le} \left(3^M \sqrt{\max_i \mathrm{Inf}_i} I + \frac{1}{m}\right) \cdot I$$

L7.3 follows from earlier results.

We've chosen M s.t. $M3^M = \frac{1}{\sqrt{\max_i \mathrm{Inf}_i}}$. We can do this, since $\sqrt{\dots} > 0$, LHS is continuous, $\lim_{M \searrow 0} m3^m = 0$ and $\lim_{m \to \infty} = \infty$.
Therefore we get $3^M \sqrt{\max_i \mathrm{Inf}_i} = \frac{1}{M}$. So $\mathrm{Var}(f) \le \frac{2}{M} I$, i.e.

$$I \ge \frac{m}{2} \mathrm{Var}(f) \tag{7.1}$$

Left to show: $m \ge \text{const} \ln \left(\frac{1}{\max_i \mathrm{Inf}_i}\right)$

Note that $6^M \ge \frac{1}{\sqrt{\max_i \mathrm{Inf}_i}}$ therefore $M > \log_6 \left(\frac{1}{\sqrt{\max_i \mathrm{Inf}_i}}\right) = \frac{1}{2\ln(6)} \ln \left(\frac{1}{\sqrt{\max_i \mathrm{Inf}_i}}\right)$
Substituting this into (7.1) we get what we want.

Corollary:
Now we want: $\exists c > 0$ s.t. $\forall n, \forall A \subseteq \{0,1\}^n, \forall 0 < p < 1$ we have

$$\sum_i \mathrm{Inf}_i(A) \geq c\mathbb{P}(A)\left(1 - \mathbb{P}(A)\right) \ln\left(\frac{1}{\sqrt{\max_i \mathrm{Inf}_i(A)}}\right)$$

Proof:

SKETCH OF PROOF Let $x_1, \ldots, x_n \in \{\pm 1\}$ iid, with $\mathbb{P}(x_i = \pm 1) = p$, with $\langle f, g\rangle = \mathbb{E}f(x)g(x)$.
The problem is that $\langle x^S, x^T\rangle \neq 1_{\{S=T\}}$ if $p \neq 2$. Observe that $\langle X^S, X^T\rangle = \mathbb{E}(x^2)^{S\cap T} \cdot X^{S\delta T} = \mathbb{E}(X^{S\delta T}) = \prod_{i\in S\Delta T} \mathbb{E}(X_i) = (2p-1)^{|S\Delta T|}$.
Therefore we have to find a different basis:
Exercise:
$\psi_S(x) = \prod_{i\in S} \phi(x_i)$ with $\phi(1) = \sqrt{\frac{1-p}{p}}, \phi(-1) = -\sqrt{\frac{p}{1-p}}$. Show that this is an orthogonal basis.

If we use this, we can follow the proof above, but for the orth. basis $\phi_S(x)$ instead of x^S to complete the proof.

proof Note that it is enough to show for p dyadic radionals. So $p = \frac{k}{2^l}$ with $k, l \in \mathbb{N}\cup\{0\}$.
Note that dyadic rationals are dense in $[0,1]$. We see that $p \mapsto \mathbb{P}_p(A)$ and $p \mapsto \mathrm{Inf}_i(A)$ are continuous in P for fixed n, A.
Therefore $p \mapsto \sum \inf, p \mapsto \max \mathrm{Inf}, p \mapsto \mathbb{P}(A)(1-\mathbb{P}(A))$.
Fix p with $p_1, p_2, \ldots$ dyad. rat. s.t. $p_n \to p$. Therefore

$$\sum \mathrm{Inf}^{(p)} = \lim \mathrm{Inf}^{p_m} \geq \lim_{m\to\infty} c\mathbb{P}_{p_m}(A)\left(1-\mathbb{P}_{p_m}(A)\right)\ln\left(\frac{1}{\max_i \mathrm{Inf}^{p_m}(A)}\right)$$
$$= c\mathbb{P}_p(A)\left(1-\mathbb{P}_p(A)\right)\ln\left(\frac{1}{\max_i \mathrm{Inf}^{p}(A)}\right)$$

Therefore it is indeed enough to show for p dyadic rationals, which we will do now:
Let $p = k/2^l, Y_1, \ldots, Y_l$ be IID Be(1/2).

$$U := \sum_{j=1}^{l} 2^{-1} Y_j \overset{d}{=} \mathrm{Uni}\left(\left\{0, \frac{1}{2^l}, \frac{2}{2^L}, \ldots, \frac{2^l - 1}{2^l}\right\}\right)$$

This follows since $(Y_1, \ldots, Y_e) \sim \text{Unif}\left(\{0,1\}^l\right)$, and last part in align, is in bijection with $\text{Unif}\left(\{0,1\}^l\right)$

$X := 1_{\{u<p\}} \stackrel{d}{=} \text{Be}(p)$. This is since $\mathbb{P}(X=1) = \mathbb{P}(U \in \{0, \frac{1}{2^l}, \ldots, \frac{k-1}{2^l}\})$. Note that we have k possiblities, since distribution of U we see that we have probability for each element of U is $\frac{1}{2}$ therefore $\mathbb{P}(X=1) = \frac{k}{2^l} = p$.

Start with $A \subseteq \{0,1\}^n$ $Y_{i,j}; 1 \le i \le n, 1 \le j \le l$ iid Be(0.5).
$X := 1_{\{\sum_{j=1}^{l} Y_{i,j} \cdot 2^{-j} < p\}} \stackrel{d}{=} \text{Be}(p)$.
Note: $B \subseteq \{0,1\}^{nl}$. B holds wrt Y_{ij} if A holds wrt $X_1, \ldots, X_n$.
Therefore talagrand for $p = \frac{1}{2}$ we get

$$\sum_{i,j} \text{Inf}_{i,j}(B) \ge c\mathbb{P}(A)(1-\mathbb{P}(A)) \ln\left(\frac{1}{\max_{i,j} \text{Inf}_{i,j}(B)}\right)$$

$\text{Inf}_{ij}(B)$ if Y_{ij} has influence then

- X_i has influence (on A)
- Flipping Y_{ij} changes whether $X_i = \sum_t Y_{it} 2^{-t} < p$.

Whether X_i pivotal does not depend on X_i but on $Y_{i',j'}; i' \neq j'$. Also depends whether Y_{ij} is pivotal for $X_i = 1$ depends only on $Y_{i',j'}$ $(i \neq j')$. Therefore

$$\begin{aligned}\text{Inf}_{ij}(B) &= \mathbb{P}(X_i \text{ pivotal for } A) \cdot \mathbb{P}(Y_{ij} \text{ pivotal for } X_i = 1) \\ &= \text{Inf}_i(A) \times \mathbb{P}(Y_{ij} \text{ pivotal for } X_i) \le \text{Inf}_i(A)\end{aligned}$$

Therefore we get

$$\sum_{i,j} \text{Inf}_{ij}(B) \ge c\mathbb{P}(A)(1-\mathbb{P}(A)) \ln\left(\frac{1}{\max_i \text{Inf}_i(A)}\right)$$

Now note that

$$\mathbb{P}(Y_{ij} \text{ pivotal for } X_i) \le \mathbb{P}\left(X_i \in [p - 2^{-j}, p + 2^{-j})\right)$$

Left as exercise that $\mathbb{P}(Y_{ij} \text{ pivotal for } X_i) \le 2^{1-j}$
Therefore we see that $\sum_{i,j} \text{Inf}_{i,j}(B) \le \sum_{i,j} \text{Inf}_i(A) \cdot 2^{1-j} \le 2\sum_i \text{Inf}_i(A)$. Therefore

we get,

$$2\sum_i \mathrm{Inf}_i(A) \geq c\mathbb{P}(A)(1-\mathbb{P}(A))\ln\left(\frac{1}{\sqrt{\max_i \mathrm{Inf}_i(A)}}\right)$$

$$\sum_i \mathrm{Inf}_i(A) \geq \hat{c}\mathbb{P}(A)(1-\mathbb{P}(A))\ln\left(\frac{1}{\sqrt{\max_i \mathrm{Inf}_i(A)}}\right)$$

Lecture 8

Recall from exercise: $\forall p \in \{0,1,2,\ldots\} \cup \{\infty\}$
$\mathbb{P}(\exists\,\text{exactly}\,k\,\infty\,\text{clusters}) = 1$
Let $N = \#$ distict infinite clusters
Lemma:
$\mathbb{P}(k \in \{0,1,\infty\}) = 1.$
Proof:
Suppose $\mathbb{P}(N = k) = 1$ for some $2 \le k < \infty$
$E_n = \mathbb{P}(N = k\&\text{each}\,\infty\,\text{cluster hits}\,\Lambda_n)$. Therefore $\bigcup_n E_n = \{N = k\}$.
So $\lim \mathbb{P}(e_n) = \mathbb{P}(N = k) = 1$. Pick n s.t. $\mathbb{P}(E_n) > \frac{1}{2}$.
Experiment:

1. Generate model
2. Resample $E(\Lambda_n)$

Outcome: original model.
$F := \{E_n\,\text{holds in stage}\,1)\,\text{and}\,E(\Lambda_n)\,\text{all open in stage}\,2)\}$. Therefore $\mathbb{P}(F) = \mathbb{P}(E_n) \cdot p^{E(\Lambda_n)} > 0$.
Now note if F happens, there is exactly 1 infinite path. This is because every edge in Λ_n is open, so connects all infinite paths to 1 infinite paths.
Note that $\mathbb{P}(N = 1) \ge \mathbb{P}(F) > 0$ but this is contradiction, since we said $\mathbb{P}(N = k) = 1$ for all $k \in [2,\infty)$. Hence we proved the lemma.

z is TRIFURCATION POINT if z has exactly 3 neighbours, and if we remove z, it's cluster splits into $3\,\infty$ clusters.
Note that $\mathbb{P}(z\,\text{tri}) = \mathbb{Z}(\underline{0}\,\text{tri}) =: q > 0$.

Lemma:
If $\mathbb{P}(N = \infty) = 1$ then $\mathbb{P}(\underline{0}\text{tri. point}) > 0$.
Proof:
$E_n = \Lambda_n$ hits at least 3 infinite clusters.
$\mathbb{P}(E_n) \xrightarrow{n\to\infty} 1.$ since $\bigcup_n E_n = \{\exists\,\text{at least 3 clusters}\}$
Therefore fix n s.t. $\mathbb{P}(E_n) > \frac{1}{2}$. If E_n holds, we can find z_1, z_2, z_3 on $\partial\Lambda_n$ distinct s.t.
$z_i \overset{\Lambda_n^c}{\leftrightsquigarrow} \infty\,\forall i$ and $z_i \not\leftrightsquigarrow z_j, \forall i \ne j$.
$F = \{E_n\,\text{holds in stage}\,1)$, fix z_1, z_2, z_3 in stage 3, $\exists 3$ disjoint paths between $\underline{0}\&z_1/z_2/z_3$ in Λ_n and all other edges in Λ_n are closed$\}$.
Therefore $\mathbb{P}(F) = \mathbb{P}(E_n)\mathbb{P}(F|E_n) \ge \mathbb{P}(E_n) \times \min(p, 1-p)^{E(\Lambda_n)} > 0$

$\mathbb{P}(\underline{0}\text{trif point}) \geq \mathbb{P}(F) > 0.$

Fix n large TBD.
$\mathcal{X} := \{z \in \Lambda_n : z \text{ trif}\}$ and $\mathcal{Y} := \{z \in \partial\Lambda_n : z \overset{\Lambda_n}{\leftrightsquigarrow} \mathcal{X}\}$
$X = |\mathcal{X}|, Y = |\mathcal{Y}|$, so $\mathbb{E}[X] = \mathbb{E}\left(\sum_{z\in\Lambda_n} 1_{\{z \text{ tri}\}}\right) = q|\Lambda_n|$
Therefore $\mathbb{P}(X \geq q|\lambda_n|) > 0$.
Claim: $X \leq Y$ (Deterministially)
$G :=$ open graph inside Λ_n.
$G \supseteq F =$ made by repeating if $\exists$ cycle, delete some edges on it.
F is a spanning forest, forest=acyclic graph=where every component is a tree.
$H \subseteq F$ by repeating if $\exists$ leaf (point of order 1) not in Y remove it and incident edge.
Note $\mathcal{Y} \subseteq V(H)$. So if $z \in \mathcal{X}$ is tri point, then $\exists y_1, y_2, y_3 \in \mathcal{Y}$ paths p_1, p_2, p_3 edges disjoints that connect z to y_1, y_2, y_3.
Also true in G and F, since you can never remove edges incident with z, since z not any cycle otherwise not trif. points.
Note that $\forall$ points on p_1, p_2, p_3 has degree at least 2 or $\in \mathcal{Y}$.
Exercise:
Show that F forest, then $|\text{leaves}| \geq |\#\text{vertices of deg } \geq 3|$.

Note that $|\Lambda_n| = (2n+1)^d = \Omega(n^d)$, and $|\partial\Lambda_n| = O(n^{d-1})$.
So $Y \leq |\partial\Lambda_n|$. With positive probability, $|X| \geq q|\lambda_n|$, which is true for all n.
But we need $X \leq Y$ so we need $q|\Lambda_n| \leq |\partial\Lambda_n||$
So we have $q \leq \frac{|\partial\Lambda_n|}{|\Lambda_n|} \xrightarrow{n\to\infty} 0$. Which implies $q = 0$, but we see taht $\mathbb{P}(N = \infty) = 1 \Rightarrow q > 0$ so $\mathbb{P}(N = \infty) = 0$.

Theorem:
$\forall d \geq 1, \forall 0 \leq p \leq 1, \mathbb{P}(\exists \geq 2 \text{ distinct } \infty \text{ open clusters}) = 0$.
Proof by Burton and Kean (deduced above)

We see by previous lecture that $\lim_{q\searrow p} \theta(q) = \theta(p)$.
Theorem:
$\forall d, p \to \theta(p)$ is continuous on $[0,1] \setminus \{p_c\}$.
Proof:
On $[0, p_c)$ trivial. Remains to show that $\lim_{q\nearrow p} \theta(q) = \theta(p), \forall p \in (p_c, 1]$.
Fix $p > p_c$ with $p_c < q_1 < q_2 < \ldots < p$ with $q_n \to p$. By AKW,$\forall n$ with probability $\exists$infinite cluster C_{p_i}. Use standard increasing coupling. So $G_{q_1} \subseteq G_{q_2} \subseteq \ldots \subseteq G_p$, so $C_{p_i} \subseteq C_{p_i+1}$.

$$\theta(p) - \lim_n \theta(q_n) = \lim_n \mathbb{P}(\underline{0} \in C_p) - \mathbb{P}(\underline{0} \in C_{q_n}) = \lim \mathbb{P}(\underline{0} \in C_p \setminus C_{q_n})$$

Note that $(\underline{0} \in C_p \setminus C_{q_n}) \supseteq (\underline{0} \in C_p \setminus C_{q_n+1})$. Therefore we get that

$$\theta(p) - \lim_n \theta(q_n) = \mathbb{P}\left(\underline{0} \in C_p \setminus \bigcup_n C_{q_n}\right)$$

Suppose $\underline{0} \in C_p, \underline{0} \notin \bigcup_n C_{q_n}$. Since $C_{q_1} \subseteq C_p$ there exists path P in C_p with edges $e_1, e_2, \ldots e_k$.
Since $P \subseteq C_p$, we see that $U_{e_1}, \ldots, U_{e_k} \leq p$.
On the other hand, $\forall n, \exists 1 \leq i \leq k$ s.t. $e_i \notin G_{q_n}$, so $p \geq \max_{i=1,\ldots,k} U_{e,i} \geq q_n, \forall n$. By taking limiits, we see that $p \geq \lim_{n\to\infty} q_n = p$.
$\{\underline{0} \in C_p \setminus \bigcup_n C_{q_n}\} \subseteq \{\exists e \text{ s.t. } U_e = p\}$
$\mathbb{P}(\exists e \in E(\mathbb{Z}^d); U_e = p) \leq \sum_e \mathbb{P}(U_e = p) = \sum_e 0 = 0$. So therefore, $\theta(p) - \lim_n \theta(q_n) = 0$.

Two events $A, B \subseteq \{0,1\}^n$.
$A \Box B = A \& B$ hold for "disjoint reasons".
$A \Box B = \{x = (x_1, \ldots, x_n) \in \{0,1\}^n; \exists \text{ disjoint } I, J \subseteq \{1, \ldots, n\} \text{ s.t. } 1_I \in A, 1_J \in B \& x \geq 1_I, \hat{x} \geq 1_J\}$
BK inequality:
$\mathbb{P}(A \Box B) \leq \mathbb{P}(A)\mathbb{P}(B)$ if A, B are up-sets.

Lecture 9

Theorem (BK)
For all $n \in \mathbb{N}, 0 \leq p \leq 1$ with $A, B \subseteq \{0,1\}^n$ up-sets, then

$$\mathbb{P}_p(A \Box B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B) \tag{9.1}$$

Proof:
Induction on n. For $n = 1$, the only upsets are $\{1\}, \{0,1\}$ so there is nothing to show. We assume that (9.1) holds for all $\hat{A}, \hat{B} \subseteq \{0,1\}$.
Let $A, B \subseteq \{0,1\}^n$ be arbitrary upsets. Define $C := A \Box B$. Let

$$\begin{aligned} A_i &:= \{(x_1, \dots, x_{n-1}) \in (0,1)^{n-1} \text{s.t.}\, (x_1, \dots, x_{n-1}, i) \in A\} \\ C_i &:= \{(x_1, \dots, x_{n-1}) \in (0,1)^{n-1} \text{s.t.}\, (x_1, \dots, x_{n-1}, i) \in C\} \\ B_i &:= \{(x_1, \dots, x_{n-1}) \in (0,1)^{n-1} \text{s.t.}\, (x_1, \dots, x_{n-1}, i) \in B\} \end{aligned}$$

Note A_i, B_i, C_i are up-sets since A, B upsets.
Furthermore $C_0 = A_0 \Box B_0$, and

$$C_1 = (A_0 \Box B_1) \cup (A_1 \Box B_0) \tag{9.2}$$

Since we want disjoint, we want that the nth edge is not simoltenously used for A, B, so therefore we can not have $A_1 \Box B_1$. $A_0 \subseteq A_1, B_0 \subseteq B_1$ since they are upsets.
Using this we can conclude that

$$C_0 \subseteq (A_0 \Box B_1) \cap (A_1 \Box B_0) \tag{9.3}$$

. This is since $(A_0 \Box B_1) \supseteq A_0 \Box B_0$ sim. for $(A_1 \Box B_0)$. We also have $C_1 \subseteq A_1 \Box B_1$.
Note that $A_0, A_1, B_0, B_1, C_0, C_1$ are of dimension $n-1$ so we can use IH.

$$\begin{aligned} \mathbb{P}(C_0) &= \mathbb{P}(A_0 \Box B_0) \overset{\text{IH}}{\leq} \mathbb{P}(A_0)\mathbb{P}(B_0) \\ \mathbb{P}(C_1) &= \mathbb{P}(A_1 \Box B_1) \overset{\text{IH}}{\leq} \mathbb{P}(A_1)\mathbb{P}(B_1) \\ \mathbb{P}(C_0) + \mathbb{P}(C_1) &\overset{(9.2),(9.3)}{\leq} \mathbb{P}\left((A_0 \Box B_1) \cap (A_1 \Box B_0)\right) + \mathbb{P}\left((A_0 \Box B_1) \cup (A_1 \Box B_0)\right) \\ &\overset{9.1}{=} \mathbb{P}(A_0 \Box B_1) + \mathbb{P}(A_1 \Box B_0) \\ &\overset{\text{IH}}{\leq} \mathbb{P}(A_0)\mathbb{P}(B_1) + \mathbb{P}(A_1)\mathbb{P}(B_0) \\ \mathbb{P}(C) &= (1-p)\mathbb{P}(C_0) + p\mathbb{P}(C_1) \\ &= (1-p)^2\mathbb{P}(C_0) + p\mathbb{P}(C_1) + p(1-p)\left(\mathbb{P}(C_0) + \mathbb{P}(C_1)\right) \\ &\leq (1-p)^2\mathbb{P}(A_0)\mathbb{P}(B_0) + p\mathbb{P}(A_1)\mathbb{P}(B_1) \\ &+ p(1-p)\mathbb{P}(A_0)\mathbb{P}(B_1) + p(1-p)\mathbb{P}(A_1)\mathbb{P}(B_1) \\ &= \left((1-p)\mathbb{P}(A_0) + p\mathbb{P}(A_1)\right)\left((1-p)\mathbb{P}(B_0) + p\mathbb{P}(B_1)\right) = \mathbb{P}(A)\mathbb{P}(B) \end{aligned}$$

9.1 follows from $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$.

Exponential decay in subcritical regime

SUBCRITICAL REGIME: $p < p_c$.

$\Theta_n(p) := \mathbb{P}_p(\underline{0} \leftrightsquigarrow \partial\Lambda_n)$.
EDGE BOUNDARY set S defined by $\partial_E(S) := \{(u,v) : e = uv \in E(\mathbb{Z}^d); u \in S, v \notin S\}$.
For $0 \leq p \leq 1 : \phi(S,p) := p \cdot \sum_{(u,v)\in\partial_E(S)} \mathbb{P}_p(\underline{0} \overset{S}{\leftrightsquigarrow} u)$.
Note that $\phi(S,p) = \underline{0}$ if $\underline{0} \notin S$.
Lemma:
If there is finite $S \subseteq \mathbb{Z}^d$ with $\underline{0} \in S$ and $\phi(S,p) < 1$. Then there is a $c = c(p,d) > 0$ s.t. $\Theta_n(p) \leq e^{-cn}, \forall n$.
Proof:
Let S be as above. and L be s.t. $S \subseteq \Lambda_{L/100}$. For $k \geq 2$

$$\begin{aligned}
\Theta_{kL}(p) &\leq \sum_{(x,y)\in\partial_E(S)} \mathbb{P}_p(\exists \text{ open path from } \underline{0} \text{ to } \partial\Lambda_k \& P \text{ leaves } S \text{ for the first time at } xy) \\
&\leq \sum_{(x,y)\in\partial_E(S)} \mathbb{P}_p(\{\underline{0} \overset{S}{\leftrightsquigarrow} x\} \square \{y \leftrightsquigarrow \partial\Lambda_{kL}\} \square \{xy \text{ open}\} \\
&\overset{\text{BK}}{\leq} \sum_{(x,y)\in\partial_E(S)} \mathbb{P}_p(\underline{0} \overset{S}{\leftrightsquigarrow} x)\mathbb{P}_p(y \leftrightsquigarrow \partial\Lambda_{kL}) \cdot p
\end{aligned}$$

If $x \in S \subseteq \Lambda_{L/100}$ and $xy \in E(\mathbb{Z}^d)$ then $y \in \Lambda_{L/10}$. Thus

$$\begin{aligned}
\mathbb{P}_p(y \leftrightsquigarrow \partial\Lambda_{kL}) &\leq \mathbb{P}(y \leftrightsquigarrow \Lambda_{(k-1)L} xy) \\
&= \mathbb{P}_p(\underline{0} \leftrightsquigarrow \Lambda_{(k-1)L})
\end{aligned}$$

Plugging back in

$$\begin{aligned}
\Theta_{kL}(p) &\leq \sum_{(x,y)\in\partial_E(S)} \mathbb{P}_p(\underline{0} \overset{S}{\leftrightsquigarrow} x)\mathbb{P}_p(\underline{0} \leftrightsquigarrow \Lambda_{(k-1)L})p \\
&= \phi(S,p)\Theta_{(k-1)L}(p)
\end{aligned}$$

If we iterate this we get

$$\Theta_{kL}(p) \leq (\phi(S,p))^k$$

- $n = kL$, then

$$\Theta_n(p) \leq e^{-c_0 n} \quad c_0 = \frac{1}{L} \ln\left(\frac{1}{\phi(S,p)}\right)$$

- $n \geq 2L$ then

$$\Theta_n(p) \leq \Theta_L \cdot \left\lfloor \frac{n}{L} \right\rfloor (p) \leq e^{-c_0} \cdot \left\lfloor \frac{n}{L} \right\rfloor \leq e^{-\frac{c_0}{2}n}$$

This follows from $L \cdot \left\lfloor \frac{n}{L} \right\rfloor \geq L\left(\frac{n}{L} - 1\right) = n - L \geq \frac{n}{2}$.

Claim: We are done, by setting. by $c(p,d) = c = \min\left(\frac{c_0}{2}, \min \lim_{1,\ldots,2L-1} \frac{1}{n} |\ln\left(\mathbb{P}_p(\underline{0} \leftrightsquigarrow \partial\Lambda_n\right)|^{-1}\right)$

Lecture 10

Lemma:
$\forall p \in (0,1)$ we have $\Theta_n'(p) = \frac{1}{p(1-p)}\mathbb{E}_p\phi(T,p)$ where $T := \{z \in \Lambda_n : z \not\leftrightsquigarrow \partial\Lambda_n\}$.
Proof:

$$\begin{aligned}
\theta_n'(p) &\overset{\text{MR}}{=} \sum_{e \in E(\Lambda_n)} \mathbb{P}_p(e \,\text{pivotal for}\, \underline{0} \leftrightsquigarrow \partial\Lambda_n) \\
&= \frac{1}{1-p} \sum_{e \in E(\Lambda_n)} \mathbb{P}_p(e \,\text{closed and pivotal for}\, \underline{0} \leftrightsquigarrow \partial\Lambda_n) \\
&= \frac{1}{1-p} \sum_{\substack{x,y \in \Lambda_n \\ xy \in E(\Lambda_n)}} \mathbb{P}_p(\underline{0} \leftrightsquigarrow x, y \leftrightsquigarrow \partial\Lambda_n, 0 \not\leftrightsquigarrow \partial\Lambda_n) \\
&= \frac{1}{1-p} \sum_{\substack{S \subseteq \Lambda_n \\ \underline{0} \in S}} \sum_{(x,y) \in \partial_E(S)} \mathbb{P}_p(\underline{0} \overset{S}{\leftrightsquigarrow} x \& T = S) \\
&= \frac{1}{1-p} \sum_{\substack{S \subseteq \Lambda_n \\ \underline{0} \in S}} \sum_{(x,y) \in \partial_E(S)} \mathbb{P}_p(\underline{0} \overset{S}{\leftrightsquigarrow} x)\mathbb{P}_p(T = S) \\
&= \frac{1}{1-p} \sum_{\substack{S \subseteq \Lambda_n \\ \underline{0} \in S}} \frac{\phi(S,p)}{p}\mathbb{P}_p(T = S) \\
&= \frac{1}{p(1-p)}\mathbb{E}_p\phi(T,p)
\end{aligned}$$

Note that $\phi(S,p) = 0$ of $\underline{0} \notin S$.

Aizenmann Barsky Menschikov Theorem:
In every dimension $d \geq 2$, for every $p < p_c(\mathbb{Z}^d)$ there is a constant $c = c(p,d) > 0$, s.t. $\mathbb{P}_p(0 \leftrightsquigarrow \partial\Lambda_n) \leq e^{-cn}$
Proof:
Set $\tilde{p}_c := \sup\{p \in [0,1], \exists, S < \infty, \underline{0} \in S \& \phi(s,p) < 1\}$
by last Lemma Lecture 9, $\forall p < \tilde{p}_c, \exists c > 0$ s.t $\Theta_n(p) \leq e^{-nc}, \forall n$.
To show: $\tilde{p}_c \geq p_c$, since then lemma L9 holds $\forall p < p_c$. Suffices to show $\Theta(p) > 0$ for all $p > \tilde{p}_c$. Assume that $\Theta(p) = 0$ for all $p > \tilde{p}_c$. Then it also holds that $\Theta(q) = 0$ for $q \leq p$. Let $\tilde{p}_c < q \leq p$ be arbitrary. By def. it holds that $\phi(s,q) \geq 1$ for

all $S \subseteq \Lambda_n$ containing $\underline{0}$. By second lemma

$$\begin{aligned}\Theta_n'(q) &= \frac{1}{q(1-q)}\mathbb{E}_q\phi(T,q) \geq \frac{1}{q(1-q)}\mathbb{E}_q\mathbf{1}_{\underline{0}\in T} = \frac{1}{q(1-q)}\mathbb{P}_q(\underline{0}\in T)\\ &= \frac{1}{q(1-q)}\left(1-\Theta_n(q)\right)\end{aligned}$$

Since subcritical we see that $0 = \lim_{n\to\infty}\Theta_n(p)$. There is a n_0 s.t, $\Theta_n(p) < \frac{1}{2}, \forall n \geq n_0$. So $\Theta_n(q) < \frac{1}{2}$ if $q \leq p$ and $n \geq n_0$.
Then for $n \geq n_0$ if we integrate below, we get

$$\Theta_n(p) \geq \Theta_n(\tilde{p}_c) + \int_{\tilde{p}_c}^{p} \frac{1}{q(1-q)}(1-\Theta_n(q))dq \geq \int_{\tilde{p}_c}^{p} \frac{1}{p(1-\tilde{p}_c)}\frac{1}{2}dq = \frac{p-\tilde{p}_c}{2p(1-\tilde{p}_c)}$$

If we take limit $n \to \infty$ on both sides, we see that

$$0 \geq \frac{p-\tilde{p_c}}{2p(1-\tilde{p}_c)} > 0$$

Which is a contradiction. So $\Theta(p) \not 0$ for all $p > \tilde{p}_c$ so $\Theta(p) > 0$ for all $p > \tilde{p}_c$.

Corollary (on proof):
For every $d \geq 2$, there is a $c = c(d) > 0$ s.t. $\Theta(p) \geq c(p-p_c)$ for all $p \geq p_c$.
Proof:
Exercise

Corollary on corollary:
Θ is not differentiable at $p = p_c$.

Theorem Russo
Dimension $d = 2, p \mapsto \Theta(p)$ is diff. on $(0,1) \setminus \{p_c\}$.
Proof:
$R := \sup_n\{n : \underline{0} \leftrightsquigarrow \partial\Lambda_n\}$ which is radius of the cluster of the origin.
$C := |\{z \in \mathbb{C}^d : \underline{0} \leftrightsquigarrow z\}|$ is volume/size of the cluster of the origin.
Lattice animal connected subgraph of $\mathbb{Z}^d$ that contains the origin.
For $p < p_c$ we see that $\Theta(p)' = 0$ so let $p > p_c$. $\theta(p) = 1 - \sum_{n=1}^{\infty}\mathbb{P}_p(c = n)$. So

$$\theta'(p) = -\left[\sum_{n=1}^{\infty}\mathbb{P}_p(c=n)\right]'$$

Let $a_{n,m,r}$ be the number of lattice animals with n vertices, m edges and r edges that have to be closed in order to form this lattice animal. With this, we can write

$$\mathbb{P}(c = n) = \sum_{m\geq 0}\sum_{r\geq 0} a_{n,m,r}p^m(1-p)^r$$

Note that since n is fixed, we have $m, r \not\to \infty$. So finite sum hence we can calculate derivative. Note that $a_{n,m,r}$ is a number so independent of p. Therefore

$$\begin{aligned}\frac{d}{dp}\mathbb{P}(c = n) &= \sum_{m\geq 0}\sum_{n\geq 0} a_{n,m,r}(mp^{m-1}(1-p)^r - rp^m(1-p)^{r-1}) \\ &= \sum_{m\geq 0}\sum_{r\geq 0} a_{n,m,r}p^m(1-p)^r\left(\frac{m}{p} - \frac{r}{1-p}\right)\end{aligned}$$

For $d = 2$, we see that $m, r \leq 4n$. Therefore

$$\begin{aligned}\left|\frac{d}{dp}\mathbb{P}(c = n)\right| &\leq \frac{4n}{p(1-p)}\sum_{m,r\geq 0} a_{n,m,r}p^m(1-p)^r \\ &= \frac{4n}{p(1-p)}\mathbb{P}(c = n)\end{aligned}$$

Fix $p_c < \alpha < \beta < 1$. Then for all $\alpha < p < \beta$, so $p \in (\alpha, \beta)$, then

$$\left|\frac{d}{dp}\mathbb{P}(c = n)\right| \leq \frac{4n}{\alpha(1-\beta)}e^{-c'\sqrt{n}}$$

(Follows from exercise 4.ii). Then since $d = 2, \forall p \in (\alpha, \beta)$

$$\left|\sum_{m\geq n}\frac{d}{dp}\mathbb{P}_p(c = n)\right| \leq \frac{4}{\alpha(1-\beta)}\sum_{n\geq m} ne^{-c'(\alpha)\sqrt{n}}$$

So if $m \to \infty$ we see that this will go to zero,by reminder theorem hence we see that $\frac{d}{dp}\Theta(p)$ exists for $p > p_c$.

Reminder theorem. If $f_1, f_2, \ldots$ are a sequence of diff functions, ,then the derivative of $\sum_{n=1}^{\infty} f_n$ exists and $\frac{\partial}{\partial x}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} f'(x)$ if $\forall\varepsilon > 0, \exists m_0 = m_0(\varepsilon)$ s.t.

$$\left|\sum_{m\geq m_0} f_m'(y)\right| < \varepsilon \quad \forall y \in I \text{ with } x \in I \text{ open interval}$$

Lecture 11

$\mathcal{C} := \{z \in \mathbb{Z}^d : \underline{0} \leftrightsquigarrow z\}$ and $R := \sup\{n : \underline{0} \leftrightsquigarrow \partial\Lambda_n\}$
$\mathbb{P}(R \geq n) \leq e^{-cn}$ if $p \leq p_c$. and $\mathbb{P}(H(R))$SEE LECTURE NOTES
$C := |\mathcal{C}|$ then $\mathbb{P}(C \geq n) \leq e^{-cn^{\frac{1}{d}}}$ where $p \leq p_c$.
Exercise:
Show: $\exists c' > 0; \mathbb{P}(C \geq n) \geq e^{-c'n}$.

Exponential decay of volume
$\forall p < p_c, \exists c = c(p) : \mathbb{P}(C \geq n) \leq e^{-cn}$.
Proof:
$V(G_k) = k \cdot \mathbb{Z}^d$. Denote $zw \in E(G_k) \Leftrightarrow (Z + \Lambda_{2k}) \cap (w + \Lambda_{2k}) \neq \emptyset \Leftrightarrow \|z - w\|_\infty \leq 4k$. So in 2 dimensions this is from $-4k$ to $4k$ (so 9 points), and from $-4k$ to $4k$. Similair in other direction, multiply those, remove origin (since counted twice). Only depend on d not on k.
Note that by reasoning 2 lines above, we see that $D = \deg(G_k) = 9^d - 1$ does not depend on k, only on d!
Note if $\underline{0} \notin \Lambda_{2k}$ and $\mathcal{C} \cap (z + \Lambda_k) \neq \emptyset$, then z is good.
If $C \geq n$ then $\exists$ animal A in G_k,of size at most $m := \frac{n - |\lambda_{2k}|}{|\lambda_k|}$.

$$\begin{aligned}
\mathbb{P}(C \geq n) &\leq \mathbb{P}(\exists \text{animal in } G_k \text{ of size } m) \\
&\leq \sum_{A \in \mathcal{A}_m} \mathbb{P}(\text{all points of } A \text{ good}) \\
&\leq \sum_A \mathbb{P}(\Lambda_k \leftrightsquigarrow \partial\Lambda_{2k})^{\alpha(A)} \\
&\leq \sum_A \mathbb{P}(\Lambda_k \leftrightsquigarrow \partial\Lambda_{2k})^{\frac{m}{D+1}} \\
\mathbb{P}(\Lambda_k \leftrightsquigarrow \partial\Lambda_{2k}) &\leq |\Lambda_k| \cdot \mathbb{P}(\underline{0} \leftrightsquigarrow \partial\Lambda_k) \\
&\leq (2k+1)^d e^{-ck} \xrightarrow{k \to \infty} 0
\end{aligned}$$

Choose k s.t. $\mathbb{P}(\Lambda_k \leftrightsquigarrow \partial\Lambda_{2k}) \leq e^{-1} 2^{-D(D+1)}$

$$\mathbb{P}(C \geq n) \leq 2^{Dm} \left(e^{-1} 2^{-D(D+1)}\right)^{\frac{m}{D+1}} = \left(\frac{1}{e}\right)^{\frac{m}{D+1}} \cdot 2^{Dm} 2^{-Dm} = e^{-\frac{m}{D+1}}$$

Now use that $m = \frac{n - |\Lambda_{2k}|}{|\Lambda_k|}$.
Exercise:
Complete the argument: show that $\exists c > 0$ s.t. $\frac{1}{D+1} m \geq cn$.

Exercise
G has deg $\leq d, v \in V(G)$ then $a_n :=$ #connvected subgraphs containing v then $a_n \leq 2^{Dn}$.

$\alpha(G) := \max\{|S| : S \subseteq V(G) \& uv \notin E(G), \forall u, v \in S\}$
Exercise
$\alpha(G) \geq \frac{|V(G)|}{D+1}$

Let $B = \{a_1, \ldots, b_1\} \times \{a_d, \ldots, b_d\}$ with $H(B) = \{a_1\} \times \{a_2, \ldots, b_2\} \times \ldots \times \{a_d, \ldots, b_d\} \overset{B}{\leftrightsquigarrow} \{b_1\} \times \{a_2, \ldots, b_2\} \times \ldots \times \{a_d, \ldots, b_d\}$.
New characterisation p_c.
Exercise:
Show k th root trick:
$A_1, \ldots, A_n$ are up-sets, then $\max_i \mathbb{P}(A_i) \geq 1 - \sqrt[n]{1 - \mathbb{P}(A_1 \cup \ldots \cup A_n)}$

Theorem:

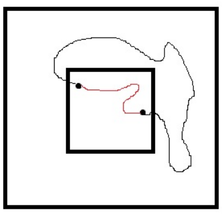
- If $p < p_c$ then $\mathbb{P}(H(\Lambda_n)) \xrightarrow{n\to\infty} 0$
- If $p > p_c$ then $\mathbb{P}(H(\Lambda_n)) \xrightarrow{n\to\infty} 1$

Proof:'

1.

$$\mathbb{P}(H(\Lambda_n)) \leq (2n+1)^d e^{-cn} \xrightarrow{n\to\infty} 0$$

2. $E_{k,n} = \{\forall z, w \in \Lambda_k : z \leftrightsquigarrow w \text{ iff } z \overset{\Lambda_n}{w}\}$ with $k \leq n$.



For any k,$\mathbb{P}(E_{k,n}) \xrightarrow{n\to\infty} 1$.
$\exists n_0 = n_0(k)$ s.t. $\mathbb{P}(E_{k,n}) \to \epsilon, \forall n \geq n_0$. Note that $\partial\Lambda_n$ has $2d$ facets. Therefore $H(\Lambda_n) = f_1 \overset{\Lambda_n}{\leftrightsquigarrow} f_2$.
$F_{k,n,i} := \{\exists z \in \Lambda_k : z \leftrightsquigarrow \infty \& z \overset{\Lambda_n}{\leftrightsquigarrow} f_i\}$ therefore $\bigcup_{i=1}^{2d} F_{k,n,i} = \{\Lambda_k \leftrightsquigarrow \infty\}$.
Can Choose k s.t. $\mathbb{P}(\{\Lambda_k \leftrightsquigarrow \infty\}) > 1 - \delta$.

By square root trick, we see that $\mathbb{P}(F_{k,n,1}) = \max_i(\mathbb{P}(F_{k,n,i}))$
$= 1 - \sqrt[2d]{(1 - \mathbb{P}(F_{k,n,1} \cup \ldots \cup F_{k,n,2d})} > 1 - 2d\sqrt{\delta} > 1 - \epsilon$ by choice of δ.
Consider $F_{k,n,1} \cap F_{k,n,2} \cap E_{k,n} \subseteq H(\Lambda_n) \cup U^c$.
Where $U = \{\exists!\text{infinite cluster}\}$.Therefore

$$\mathbb{P}(H(\Lambda_n)) \geq \mathbb{P}(F_{k,n,1} \cap F_{k,n,2} \cap E_{k,n}) - \mathbb{P}(U^c) > 1 - 3\epsilon$$

$\mathbb{P}(U^c) = 0$ by Second last theorem page 30.
take $\epsilon \searrow 0$.

critical probability in dimension 2
$\mathbb{P}_{\frac{1}{2}}(H(\{1, \ldots, m+1\} \times \{1, \ldots, m\}) = \frac{1}{2}$.
$\mathbb{P}(H(\Lambda_n)) = \mathbb{P}(H(\{1, \ldots, 2n+1\}^2)) \geq \mathbb{P}(H(\{1, \ldots, 2n+2\} \times \{1, \ldots, 2n+1\}) = \frac{1}{2} \not\to 0$.
Therefore $p_c \leq \frac{1}{2}$.
$\frac{1}{2} = \ldots \geq \mathbb{P}_{\frac{1}{2}}(H(\{1, \ldots, 2n+1\}^2)) \cdot \frac{5}{8} = \mathbb{P}(H(\Lambda_n)) \cdot \frac{5}{8}$.
Status of "these" edges indep. of event $H(\Lambda_n)$. Therefore we see that if we have a horizontal crossing in $2n \times 2n+1$ box, then there is a possibility we end in a corner. Then we have a probability of $\frac{5}{8}$ to pass the entire box. (namely $\frac{1}{2} + \frac{1}{8}$)
$\mathbb{P}(H(\{1, \ldots, 2n+2\} \times \{1, \ldots, 2n+2\}) \geq \mathbb{P}(H(\Lambda_1)) \cdot \left(\frac{1}{2} + \frac{1}{8}\right)$.
So $\mathbb{P}(H(\Lambda_n)) \not\to 1$ since $\frac{1}{2} \not\geq \frac{5}{8}$. So $p_c \geq \frac{1}{2}$.

Lecture 12

T triangular lattice integer linear combination of $v_1 = (1,0)$ and $v_2 = (0.5, 0.5\sqrt{3})$.
H dual honeycomb lattice. Tutorial: Show that this is indeed a lattice.
Tutorial:

1. In T or H at most one infinite cluster (when $p < p_c(T)$)
2. $\mathbb{P}_p(\underline{0} \leftrightsquigarrow T \setminus [-n,n]^2) \leq e^{-cn}$ with $p < p_c(H)$ and $\mathbb{P}_{..}(\underline{0}_H \leftrightsquigarrow H \setminus [-n,n]^2) \leq e^{-cn}$.
3. $R_n = \{iv_1 + jv_2; i,j \in \{1,\dots,n\}\}$, $H(R_n)$ horizontal crossing, $H(R_n)$ in T or $V(R_n)$ in H under standard coupling.
   For $T \& H$

$$\mathbb{P}_p(H(R_n)) \to \begin{cases} 0 \text{ if } p < p_c \\ 1 \text{ if } p > p_c \end{cases}$$

Corollary:
$p_c(T) + p_c(H) = 1$.
Proof:

$$\begin{aligned} 1 &= \mathbb{P}_p(H(R_n) \text{ in } T) + \mathbb{P}_{1-p}(V(R_n) \text{ in } H) \\ &= \mathbb{P}_p(H(R_n) \text{ in } T) + \mathbb{P}_{1-p}(H(R_n) \text{ in } H) \end{aligned}$$

Pick $p > p_c(T)$, therefore $\mathbb{P}_p(H(R_n) \text{ in } T) \to 1$, and $\mathbb{P}_{1-p}(H(R_n) \text{ in } V) \to 0$. Therefore $1 - p \leq p_c(H)$, so $1 - p_c(T) \leq p_c(H)$.
Pick $p < p_c(T)$, therefore $\mathbb{P}_p(H(R_n) \text{ in } T) \to 0$ and $\mathbb{P}_{1-p}(H(R_n) \text{ in } H) \to 1$. So $1 - p \geq p_c(H)$. so $1 - p_c(T) \geq p_c(H)$.
So we have $p_c(H) \leq 1 - p_c(T) \leq p_c(H)$ and therefore $p_c(H) = 1 - p_c(T)$ so $p_c(H) + p_c(T) = 1$.

$\Delta - y$ TRANSFORMATION
SEE NOTES. For percolation, matters how the triangles, resp Y, connects x, y, z.

Partition	Δ	Y
$\{x,y,z\}$	$p^3 + 3p^2(1-p)$	$(1-p)^3$
$\{\{(x,y)\},\{z\}\}$	$p(1-p)^2$	$(1-p)^2 p$
$\{\{(x,z)\},\{y\}\}$	$p(1-p)^2$	$(1-p)^2 p$
$\{\{(y,z)\},\{x\}\}$	$p(1-p)^2$	$(1-p)^2 p$
$\{\{x\},\{y\},\{z\}\}$	$(1-p)^3$	$p^3 + 3(1-p)p^2$

If $\Delta = Y$, then $\Delta - Y$ transformation does not change $\mathbb{P}$(percolation), observe that we only need to hold therefore that $p^3 + 3p^2(1-p) = (1-p)^3$.
Claim: this means that $p^3 - 3p + 1 = 0$.
Exercise:

1. Unique solution in $[0,1]$ is $p = 2\sin(\pi/18)$.

2. $\sin(\pi/18) = \sqrt{2-\sqrt{2+...}}$

Applying $\Delta - Y$ transformation to all upward triangles in T, we get a coupling of T (with p), and H (with $1-p$). Note that we have shifted H.

Exercise:
$\mathbb{P}_p(H(R_n)\,\text{in}\,T) \geq \mathbb{P}_{1-p}(H(R_n)\,\text{in}\,H) \geq \mathbb{P}_p(H(R_n)\,\text{in}\,T)\cdot(1-p)$.

Result about exact value p_c by Wierman 1981
Suppose $p := 2\sin(\pi/8) > p_c(T)$, therefore $\mathbb{P}_p(H(R_n)\,\text{in}\,T) \to 1$ by an exercise we show that $\mathbb{P}_{1-p}(H(R_n)\,\text{in}\,H) \not\to 0$ so $1-p \geq p_c$ but this means that $1 = p + 1 - p > p_c(T) + p_c(H) = 1$, which is a contradiction.
Suppose $p < p_c(T)$, so $\mathbb{P}_p(H(R_n)\,\text{in}\,T) \to 0$ so $\mathbb{P}_{1-p}(H(R_n)\,\text{in}\,H) \not\to 1$, therefore $1-p \leq p_c(H)$ so $= p + (1-p) < p_c(T) + p_c(H) = 1$. which is a contradiction.
Therefore we get $p = p_c(T)$ so $p_c(T) = 2\sin(\pi/18)$ and $p_c(H) = 1 - 2\sin(\pi/18)$.

Lecture 13

Galton-Watson Trees
Need to show: $\forall \epsilon, \exists d_0 = d_0(\epsilon)$ s.t. $p = \frac{1+\epsilon}{2d}, d \geq d_0$ then $\mathbb{P}(\underline{0}$ is in ∞ open cluster$) > 0$

$x = \#$children, $x_n = 0$. Note that # children of each indiv. are iid NV $X_1, X_2, \ldots \overset{d}{=} X$. If we assume a birth under natural (unrespondence iwth iid seq.) x_1, x_2, x_3. What is $\mathbb{P}($ tree dies out$)$ so tree is ifnite.
Theorem:
If $0 < \mathbb{P}(X = 0) < 1$ then $\mathbb{P}(\text{extinction}) = 1$ iff $\mathbb{E}[X] \leq 1$.
Proof: Stochastic processes.
Exploration procedure: Find children of root-continue process previously unprocessed mode (add its children) (stop if none such exists so T finite.) So # new individuals in each step $\overset{d}{=}$ and independent of past.

K-array tree: every node has k children. So root has degree k and anyone else has degree $k + 1$. (Has to be ∞ large). Notation: T_k.
Exercise:
Show that $p_c(T_k) = \frac{1}{k-1}$.

3-regular tree: every node has degree 3.
Exercise:
$N = \#\infty$ clusters. Show that $\mathbb{P}(N = \infty) = 1$ if $p_c < p < 1$ on T_k.

Stochastic domination:
$X \geq Y$ if $\mathbb{P}(X \geq x) \geq \mathbb{P}(Y \geq x), \forall x \in \mathbb{R}$.
Exercise:
$\text{Bi}(n, p) \geq \text{Bi}(m, q)$ if $n \geq m, p \geq q$.
$X \sim \text{Be}(p), Y \sim \text{Be}(y)$, then $\mathbb{P}(X \geq 0) - \mathbb{P}(y \geq 0) = 1$,and $\mathbb{P}(x \geq 1) = p \geq q \geq \mathbb{P}(y \geq 1)$.

coupling of X, Y is probability space/random vector (X', Y') s.t. $X' \overset{d}{=} X, Y' \overset{d}{=} Y$.
Exercise $X, Y \sim \text{Be}(1/2)$.

Simple case of Strassew's theorem:
x, y integer valued, $X \underset{st}{\geq} Y$ then $\exists$ coupling (X', Y') s.t. $\mathbb{P}(X' \geq Y') = 1$.
Proof:
Exercise

$X_n = \mathrm{Bi}(n, \min(1, c/n))$ then as $n \to \infty$ then $X_n \sim \mathrm{Poi}(c)$ so $\mathbb{P}(X = k) = \frac{c^k}{k!}e^{-c}$.
Exercise:
Show that $\exists n_0 = n_0(c), y$ s.t. $X_n \underset{ST}{\geq} Y, \forall n \geq n_0$ and $\mathbb{E}[Y] > 1$.

Exercise $X \underset{ST}{\geq} Y$ with T offspring distribution X, and S offspring distribution Y (offspring = nakomelingen) then $|T| \underset{ST}{\geq} |S|$

Exercise $X_n \overset{d}{=} \mathrm{Bi}(n, p_n)$ for some sequence $(p_n)_n$ and T_n galton-watson offspring distriubtion X_n, then $\liminf E(X_n) > 1 \Rightarrow \liminf \mathbb{P})(T_n = \infty) > 0$.

Lemma:
$\forall \varepsilon, \exists N$ s.t. $\forall M, \exists d_0 = d_0(\epsilon^n, N, M)$ s.t. if $d \geq d_0, p \geq \frac{(1+\epsilon)}{2d}, \forall z_1, z_N \in \mathbb{Z}^d$ distinct then $\mathbb{P}(\max_{i=1,\dots,N} |\mathcal{C}(z_i)| < m) < \eta$, where $\mathcal{C}(z_i)$ is open clusters of z_i.
Idea: Embeded G.W. Tree: Let $T_i \subseteq \mathcal{C}(z_i)$. Want them to behave like indep. GW trees. Want:

$$\mathbb{P}(|T_1|, \dots, |T_N| < M) = \mathbb{P}(|T_1| < m)^N \tag{L13.1}$$

Define exploration procedure start from z, (first vertex of T_1) and open neighbours of z_1.
So at each step we pick unprocessed point of T_1 and add neighbours not yet in T_1 that are distinct from $z_2, \dots, z_N$. Here we still assume $|T_1| < m$. At each step # neighbours we can add at least $2d - (m-1) - (N-1)$. Note that # neighbours $\underset{ST}{\geq}$ Bi(..., p).
If we fail to build of size $\geq m$, restart building T_2 from z_2. At each step we ask for neighbours not in T_1, T_2 so far and $z_3, \dots, z_N$. Therefore noumber of possible neighbours we can add $2d - 2(M-1) - (N-2)$. Continue like this until T_N. When building T_i number of possible Neighbours is at least $2d - NM \geq \lceil (1-\delta) \cdot 2d \rceil =: n$. We see that for T_i the number of possible neighbours you can add is at least $2d - i(m-1) - (N-i)$ so if you want that it works for all i, you pick the larges, so $i = N$. Therefore we see that the number of possilbe neighbours you can add is at least $2d - N(m-1) = 2d - Nm + N$. Note that you can also add therefore $2d - NM$, with which he works.

Minor change to procedure at each step chose exactly n potential neighbours, quiz only them. Now at all steps $\#\{\text{new points}\} \overset{d}{=} X = \mathrm{Bi}(n, p)$.
$\mathbb{P}(|T_1| < m)$. If T is GW tree with distance X, then $\mathbb{P}(|T_1| < m) = \mathbb{P}(|T| < m)$.
Some inequalityTherefore we get $\mathbb{P}(|T_1|, \dots, |T_n| < m) = \mathbb{P}(|T| < m)^n \leq \mathbb{P}(|T| < \infty)^N$.
By choice of δ, we can have $(1-\delta)(1+\epsilon) > 1$ so $\mathbb{E}[X] = n \cdot p > (1-\delta) \cdot 2d \cdot (1+\epsilon) > 1$.
So by exercises $\exists c = c(\epsilon, \delta)$ not on , d s.t.$\mathbb{P}(|T| = \infty) > c$.

Implementing above, we get $\mathbb{P}(|T_1|, \dots, |T_n| < m) < (1-c)^N < \eta$—,by choice of $N = N(\epsilon, \eta)$.
$\mathbb{P}(\max_i |\mathcal{C}(z_i)| > m) \geq \mathbb{P}(\max_i |T_i| \geq m) > 1 - \eta$.
Note that this finishes the proof, since $\mathbb{P}(\max_i |\mathcal{C}(z_i)| < m) = 1 - \mathbb{P}(\max_i |\mathcal{C}(z_i)| > m) < 1 - (1-\eta) = \eta$.

Lecture 14

$\exists N$ s.t. $\forall m, \exists d_0$ s.t. $d \geq d_0$ and $p \geq (1+\epsilon)/2d$ then $\mathbb{P}(C(z_1), \ldots, C(z_n) < m) < \epsilon$
Here $C(Z) = \left|\{w \in \mathbb{Z}^d : z \leftrightsquigarrow w\}\right|$. For p take $p = 0.99$.
Main idea for Kesten's proof of GKHS. Coupling with 2d site percolation. Fix ϵ, δ and def. $m := \lfloor \delta d \rfloor$ and $\pi : \mathbb{Z}^d \to \mathbb{Z}^2$ s.t. $(x_1, \ldots, x_d) \mapsto (x_1 + \ldots + x_m, x_{m+1}, \ldots, x_{2m})$
Fix $z_1^{(0,0)}, \ldots, z_N^{(0,0)} \in \pi^{-1}(0,0)$, all distinct. $c(z, A) = \left|\{w \in A | z \overset{A}{\leftrightsquigarrow} w\}\right|$
We say $(0,0)$ is open if

1. one of $C(z_i^{(0,0)}, \pi^{-1}(0,0)) \geq M$
2. $\forall (x,y) \in \{(-1,0),(0,-1),(1,0),(0,1)\}$ exists at least N points in $\pi^{-1}(x,y)$ connected to $\cup C(z_i^{(0,0)}, \pi^{-1}(0,0))$ via an open edge.

Expolartion procedure:

- Process a node (x,y), adjecent to node which is open (previously determined). We do not process a node more than once. Until we no longer?
- We can fix $z_1^{(x,y)}, \ldots, z_N^{(x,y)}$ s.t. connected to $\{z_1^{(0,0)}, \ldots, z_N^{(0,0)}\}$ via open paths (in $\mathbb{Z}^d$) that project onto open sites of $\mathbb{Z}^d$.

Therefore we say (x,y) is open if

1. one of $C(z_i^{(x,y)}, \pi^{-1}(x,y)) \geq M$
2. $\forall (x,y) \in \{(x-1,y),(x,y-1),(x+1,y),(x,y+1)\}$ exists at least N points in $\pi^{-1}(x,y)$ connected to $\cup C(z_i^{(x,y)}, \pi^{-1}(x,y))$ via an open edge.

Want to show: N, M chosen well, $\mathbb{P}(\text{Percolates}) > 0$ so $d-$ dimensional percolation.
Enough to show: at each iteration, $\mathbb{P}(\text{current open}|\text{past}) \geq p > p_c(\mathbb{Z}^2, \text{site})$
Since this shows:
$\mathbb{P}(\text{in our 2d model percolation}) \geq \mathbb{P}_p((0,) \leftrightsquigarrow \infty \text{ in stand. site perc. model}) > 0$.
When we process (x,y) ,no edges in $\pi^{-1}(x,y)$ have been "Revealed".
Note $\pi^{-1}(0,0) \sim \mathbb{Z}^{d-2m}$. So $\mathbb{P}(\text{one of } (C(z_i^{(0,0)}, \pi^{-1}(0,0))) \text{ for some } i \text{ has at least size } m) \geq 1 - \epsilon$
Assume $2(d-2m)p > 1 + \epsilon'$ by previous lemma. Therefore
$2(d - 2\lfloor \delta d \rfloor)\frac{1+\epsilon}{2d} \geq (1-2\delta)(1+\epsilon) > 1 + \frac{\epsilon}{2}$ by choosing δ appropriate. ?
If $(x,y) \neq (0,0)$ then $\pi^{-1}(x,y)$ is disjoint union of copies of $\mathbb{Z}^{d-2m}$ namely $\Gamma_1, \ldots, \Gamma_k$.
Suppose $z_1, \ldots, z_{i_1} \in \Gamma_1, \ldots, z_{i_{k-1}}, \ldots, z_N \in \Gamma_k$. and $C(z_1, \Gamma_1)$ and $C(z_n, \Gamma_k)$ indepedent.

$$
\begin{aligned}
&\mathbb{P}(\text{each of } z_1, \ldots, z_N \text{ has} < M \text{ points in cluster of } \pi^{-1}(x,y)) \\
\geq &\mathbb{P}(\text{each of } z_1, \ldots, z_{1_i} < M \text{ in } \Gamma_1) \times \ldots \times \mathbb{P}(\text{each of } z_{i_{k-1}}, \ldots, z_N < M \text{ in } \Gamma_K) \\
\geq &(1-c)^{i_1} \ldots (1-c)^{N - i_{k-1}}
\end{aligned}
$$

$c > 0$ approx GW surv. prob. Here $(1-c)^N < \epsilon$. (So this is the probability that the node is not good, and we want the complement.

Also want:
Lemma:
$w_1, \ldots, w_M \in \pi^{-1}(x,y)$. $\mathbb{P}(w_1, \ldots, w_M$ have at least N nbs in $\pi^{-1}(x+1,y)) \geq 1-\epsilon$.
For M chosen approp. $(M = M(N,\epsilon))$.

For each $w_1, \ldots, w_M$ exists m potentail neighbours in $\pi^{-1}(x+1,y)$ then #relevant edges $e(A,B) = M \times m$ where $A = \{w_1, \ldots, w_M\}$ and $B = \pi^{-1}(x+1,Y)$ then $X \overset{d}{=} \text{Bi}(M \times m, p)$
$\mathbb{E}[X] = Mmp = M \cdot \lfloor \delta d \rfloor \times \frac{H\epsilon}{2d} \geq \frac{M\delta(1+\epsilon)}{2}$ which we make large by choosing M.
$\text{Var}(X) = Mmp(1-p) \leq \mathbb{E}X$. So
$\mathbb{P}(X < \frac{\mathbb{E}x}{2}) \leq \mathbb{P}\left(|X - \mathbb{E}X| > \frac{\mathbb{E}X}{2}\right) \leq \frac{\text{Var}(X)}{(\mathbb{E}[X]/2)^2} \leq \frac{2\mathbb{E}[X]}{(\mathbb{E}[X]^2} < \epsilon/1000$ which we can make small.
So $\mathbb{P}(X > 1000N) > 1 - \epsilon/1000$

$Y := \#\{$points in $\pi^{-1}(X+1,Y)$ connected to at least 2 of $w_1, \ldots, w_N\}$.
$\mathbb{E}Y \leq \binom{M}{2} \times m \times p^2$ here p^2 since we need 2 edges to be open.
Therefore $\mathbb{E}Y \leq M^2 \times \delta d \cdot \frac{(1+\epsilon)^2}{4d^2} \xrightarrow{d \to \infty} 0$. Therefore $\mathbb{P}(Y \geq 1) \leq \mathbb{E}Y \leq \frac{\epsilon}{1000}$.
So $\mathbb{P}(\exists$ ast least $100N$ diff. nbs$) > 1 - \frac{\epsilon}{500}$
Therefore $\mathbb{P}((x,y)$ pen$) = 1 - \{$condition 1$\} - \{$condition 2$\} + 1 - \epsilon - 4 \cdot \frac{\epsilon}{500} > .99$.

Slab $\mathbb{Z}^2 \times \{1, \ldots, n\}^{d-2} = S_n$ n-th slab.
$p_c(S_n) \geq p_c(\mathbb{Z}^d)$. Show $\lim_{n \to \infty} p_c(S_n) = p_c(\mathbb{Z}^d)$.
$\theta(p_c) = 0$ for $d = 2$ or $d \geq 11$. Tassion sibovicius $\theta(p_c, S_n) = 0, \forall n$.

Lecture 15

Grimmett Marstrand theorem:
$\forall p$ we have $p_c(\mathbb{Z}^d) = \lim_{n\to\infty} p_c(\mathbb{Z}^2 \times \{1,\ldots,n\})$.
$p = p_c + \epsilon$ then exists N s..t. $\mathbb{Z}^2 \times \{1,\ldots,n\}$ percolation.
$d = 3, n \geq m \geq 1$ then $\mathbb{P}_{p+1}(\Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} \Lambda_n) > 1-\epsilon$.
$\mathbb{P}_{p_c+\epsilon}(\Lambda_n \overset{\Lambda_n}{\leftrightsquigarrow} Q) > 1-\epsilon$.
n th root trick $A_1,\ldots,A_n$ upsets, then $\max \mathbb{P}(A_i) \geq 1 - \sqrt[n]{\mathbb{P}(A_1 \cup A_n)}$

Let $A_i = \{\Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} Q_i\}$ so $\mathbb{P}(A_i) \geq 1 - \sqrt[24]{1-\mathbb{P}(\cup A_i)} = 1 - \sqrt[24]{1-\mathbb{P}(\Lambda_n \overset{\Lambda_n}{\leftrightsquigarrow} \partial\Lambda_n)}$.
$\epsilon" = \sqrt[24]{\epsilon'}$.
$\mathbb{P}(\#v \in G : \Lambda_m \overset{\Lambda-n}{\leftrightsquigarrow} v \geq K) \geq 1-\epsilon'''$.
Suppose $\mathbb{P}(\#v \in G : \Lambda_m \overset{\Lambda-n}{\leftrightsquigarrow} v \geq K) \geq> \epsilon"'$, then by Harris $\mathbb{P}(\#v \in \partial\Lambda_n : \Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} v < 24k) > (\epsilon"')^{24}$
Let $A = \{\#v \in \partial\Lambda_n : \Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} v\}$ then $\mathbb{P}(\Lambda_m \not\leftrightsquigarrow \partial\Lambda_{n+1}) \geq \mathbb{P}(A)\cdot(1-p)^{24k\cdot 3}$. So

$$\epsilon_5 > \mathbb{P}(\Lambda_n \not\leftrightsquigarrow \Lambda_n) > (\epsilon"')^{24}(1-p)^{72k}$$

Which can give a contradiction by choices of $\epsilon"', \epsilon_5$ by right choice of m, n.
therefore $\mathbb{P}(\#v \in Q : \Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} v < K) < \epsilon"'$ therefore $\mathbb{P}(\#v \in Q : \Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} v \geq K) \geq 1-\epsilon$.

SEED copy of Λ_m with all edges open.
$\mathbb{P}(E = \{\exists v \in Q : \Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} v, v(v+e_1)\text{ open}, (v+e_1)+\{0,\ldots,2m\}^3\text{seed})\} > 1-\epsilon\delta$.
If $F = \{\#v \in Q : \Lambda_m \overset{\Lambda_n}{\leftrightsquigarrow} v \geq k\}$ then $M = \frac{K^2}{(2m+1)^2}$
Can choose potential seed disjoint K large $\Rightarrow M$ large. $\mathbb{P}(E|F) \geq 1-(1-p^{1+(2m+1)^3})^m \geq 1-\epsilon 6$ by choice of k.
AND THEN HE WENT UP TILL WITH SUCH A REASONING TILL ϵ_{10}, AND NO IDEA WHAT HE DID